Published by Institute of Physics Publishing for SISSA/ISAS
Received: *October 13, 2004*
Accepted: *November 19, 2004*

DLCQ string spectrum from $\mathcal{N} = 2$ SYM theory

Giuseppe De Risi

Dipartimento di Fisica and Sezione I.N.F.N., Università di Bari
Via G. Amendola 173, 70126 Bari, Italia
E-mail: derisi@ba.infn.it

Gianluca Grignani[*] **and Marta Orselli**[†]

Dipartimento di Fisica and Sezione I.N.F.N., Università di Perugia
Via A. Pascoli I-06123, Perugia, Italia
E-mail: grignani@pg.infn.it, orselli@pg.infn.it

Gordon W. Semenoff[‡]

Pacific Institute for Theoretical Physics, and
Department of Physics and Astronomy, University of British Columbia
6224 Agricultural Road, Vancouver, British Columbia V6T 1Z1 Canada
E-mail: gordonws@physics.ubc.ca

Abstract: We study non planar corrections to the spectrum of operators in the $\mathcal{N} = 2$ supersymmetric Yang Mills theory which are dual to string states in the maximally supersymmetric pp-wave background with a *compact* light-cone direction. The existence of a positive definite discrete light-cone momentum greatly simplifies the operator mixing problem. We give some examples where the contribution of all orders in non-planar diagrams can be found analytically. On the string theory side this corresponds to finding the spectrum of a string state to all orders in string loop corrections.

Keywords: Penrose limit and pp-wave background, AdS-CFT and dS-CFT Correspondence.

[*]Work supported in part by INFN and MURST of Italy.
[†]Work supported in part by INFN and MURST of Italy.
[‡]Work supported in part by NSERC of Canada and the PIMS String Theory CRG.

http://jhep.sissa.it/archive/papers/jhep112004053/jhep112004053.pdf

JHEP11(2004)053

Contents

1. Introduction

The AdS/CFT correspondence asserts an exact duality between a ten-dimensional type-IIB superstring theory on $AdS_5 \times S^5$ background and $\mathcal{N}=4$ supersymmetric Yang-Mills theory in flat four dimensional Minkowski spacetime [1, 2, 3]. Though it has many spectacular successes, it is still a conjecture and it is not yet clear whether it is an exact correspondence, or is only valid in some limits of the two theories. Given its potential importance as a quantitative tool for strongly coupled gauge and string theory, it is important to check it wherever possible.

AdS/CFT is a strong coupling — weak coupling duality. This makes it powerful, as it can be used to compute the strong coupling limit of either theory using the weak coupling limit of the other. On the other hand, it makes it difficult to check since it is not easy to find situations where approximate computations in both theories have an overlapping domain of validity. Early exceptions to this were some quantities which were protected by supersymmetry and didn't depend on the coupling constant at all [4], or quantities determined by anomalies which had a trivial dependence [5, 6] and a few others related to circular Wilson loops [7]–[10] which had a nontrivial dependence on the coupling constant and which could be computed for all values of the coupling.

More recently it has been realized that some large quantum number limits yield domains where accurate computations in both gauge theory and string theory could be done and compared directly with each other. The first and most powerful of these is the BMN limit. It began with the observation [11, 12] that the Penrose limit of the $AdS_5 \times S^5$ metric and 5-form field strength of the string background are the maximally supersymmetric pp-wave metric and a constant self-dual 5-form

$$ds^2 = -4dx^+dx^- + \sum_{i=1}^{8} dx^i dx^i - \sum_{i=1}^{8} (x^i)^2 dx^+ dx^+ \tag{1.1}$$

$$F_{+1234} = F_{+5678} = \text{const.} \tag{1.2}$$

respectively. Then Metsaev [13] found an exact solution of the non-interacting type-IIB Green-Schwarz string in the background (1.1) and (1.2). Shortly afterward, Berenstein, Maldacena and Nastase (BMN) [14] noted that one could take a similar limit of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory by considering states with large R-charge. They identified the Yang-Mills operators (called BMN operators) corresponding to the free string states on the pp-wave background.

The AdS/CFT correspondence predicts that the spectrum of scaling dimensions and charges under R-symmetry of these operators in the 't Hooft planar limit [15] of Yang-Mills theory should match the free string spectrum. The leading order Yang-Mills theory computation in ref. [14] and a two loop calculation in ref. [16] showed beautiful agreement.

Non-planar corrections to the spectrum of operators in Yang-Mills theory should correspond to string loop corrections in string theory. The question of non-planar corrections to the spectrum of BMN operators was considered in refs. [17] and [18]. It was found that, once operator mixing by non-planar graphs was resolved [19, 20, 21], Yang-Mills theory could be used to make predictions for the spectrum of string states on the pp-wave background. There are still ongoing attempts to check these predictions on the string side of the correspondence using light-cone string field theory [22]–[32]. Success or failure of this matching would be a highly nontrivial test of the AdS/CFT correspondence at the level of interacting strings. There have also been interesting non trivial tests of the AdS/CFT correspondence in the BMN limit at finite temperature [33].

In this paper, we shall discuss a generalization of the BMN limit that was found by Mukhi, Rangamani and Verlinde [34]. They showed that a certain limit of an orbifold of $AdS_5 \times S^5$ gives the plane-wave geometry (1.1) and (1.2) with the additional feature that the null coordinate x^- is identified periodically $x^- \sim x^- + 2\pi R^-$. The result is a discrete light-cone quantized string theory on the plane wave background. This is a generalization

JHEP11(2004)053

of the BMN limit where the light-cone momentum is discrete and there are wrapped states. The gauge theory which is dual to the orbifolded string theory is an $\mathcal{N}=2$ superconformal Yang-Mills theory. The operators of Yang-Mills theory which are dual to the string states were identified in ref. [34]. Some checks that loop diagrams in planar Yang-Mills theory reproduce the correct spectrum and a number of generalizations to other types of orbifolds and limits to obtain other compactifications of the pp-wave were considered in ref. [35].

We shall study non-planar corrections to the spectrum of the operators in the $\mathcal{N}=2$ supersymmetric Yang-Mills theory which are dual to single and multi-string states. Our main observation is that, the existence of a positive definite, discrete light-cone momentum greatly simplifies the operator mixing problem. In Yang-Mills theory, string interactions show up in the mixing by non-planar diagrams of single trace and multi-trace operators, the counterparts of single and multi-string states. Since, in the case that we shall consider, the light-cone momentum must be conserved and is discrete and positive, the number of operators with different traces which can mix at any level turns out to be finite and diagonalizing their mixing exactly is a finite problem. We use this observation to give some examples where the contribution of all orders in non-planar diagrams can be found analytically. On the string side, this corresponds to finding the spectrum of a string state to all orders in string loop corrections (and, since we are doing Yang-Mills perturbation theory, to leading orders in world-sheet momenta).

In the next section, we fix the notation and give a brief review of ref. [34].

2. Preliminaries

In this section, we will first describe the gauge and string theories which are dual to each other. Then we will discuss the Penrose limit of the string theory and the equivalent double scaling limit of the gauge theory. We will discuss the holographic dictionary of non-interacting string theory states and single trace operators in the gauge theory.

2.1 $\mathcal{N}=2$ from $\mathcal{N}=4$

The four-dimensional $\mathcal{N}=2$ supersymmetric Yang-Mills theory that we are interested in can be obtained from its parent $\mathcal{N}=4$ theory by an orbifold projection [36, 37]. We begin with $\mathcal{N}=4$ with a $\mathrm{U}(MN)$ gauge group. The orbifold group will be the cyclic group Z_M whose generator γ acts on the six scalar fields of $\mathcal{N}=4$ theory as

$$\gamma:\left(\frac{\phi^1+i\phi^2}{\sqrt{2}},\frac{\phi^3+i\phi^4}{\sqrt{2}},\frac{\phi^5+i\phi^6}{\sqrt{2}}\right)=\left(\omega\frac{\phi^1+i\phi^2}{\sqrt{2}},\omega^{-1}\frac{\phi^3+i\phi^4}{\sqrt{2}},\frac{\phi^5+i\phi^6}{\sqrt{2}}\right),\quad \omega=e^{\frac{2\pi i}{M}}\,. \tag{2.1}$$

At the same time, this transformation is embedded into the gauge group as the group element

$$g=\begin{pmatrix}1&0&0&\cdots&0\\0&\omega&0&\cdots&0\\0&0&\omega^2&\cdots&0\\0&0&0&\cdots&\cdots\\0&0&0&\cdots&\omega^{N-1}\end{pmatrix} \tag{2.2}$$

where each block of this $N \times N$ matrix is tensored with the $M \times M$ unit matrix. Then, some components of the $MN \times MN$ matrix fields are set to zero so that the equation

$$\Psi = g\,(\gamma : \Psi)\, g^\dagger$$

is satisfied, for all fields Ψ of the $\mathcal{N} = 4$ theory. The resulting $\mathcal{N} = 2$ theory has residual R-symmetry $\mathrm{U}(1) \times \mathrm{SU}(2)$ and gauge group

$$\mathrm{U}(N)^{(1)} \times \mathrm{U}(N)^{(2)} \times \cdots \times \mathrm{U}(N)^{(M)}\,. \tag{2.3}$$

$\mathrm{U}(N)^{(M+1)}$ is identified with $\mathrm{U}(N)^{(1)}$.

The resulting field content is as follows:

- M vector multiplets

$$(A_{\mu I}, \Phi_I, \psi_I, \psi_{\Phi I})\,, \qquad I = 1, \ldots, M\,. \tag{2.4}$$

  Φ_I is a complex scalar field and the Weyl fermion $\psi_{\Phi I}$ is its superpartner. A_I^μ is the gauge field and ψ_I is the gaugino. All of these fields transform in the adjoint representation of $\mathrm{U}(N)^{(I)}$.

- M bi-fundamental hypermultiplets which, in $\mathcal{N} = 1$ notation, are

$$(A_I, B_I, \chi_{AI}, \chi_{BI})\,. \tag{2.5}$$

  The complex scalar field A_I and its super-partner ψ_{AI} transform in the $(N_I, \bar{N}_{I+1})$ representation of $\mathrm{U}(N)^{(I)} \times \mathrm{U}(N)^{(I+1)}$. The pair B_I and χ_{BI} transform in the complex conjugate representation $(\bar{N}_I, N_{I+1})$.

The $\mathcal{N} = 2$ action can be found from the $\mathcal{N} = 4$ theory. The euclidean lagrangian density of $\mathcal{N} = 4$ is

$$\mathcal{L} = \frac{1}{g_{\mathrm{YM}}^2} \mathrm{TR} \left(\frac{1}{2} F_{\mu\nu} F_{\mu\nu} + D_\mu \phi^i D_\mu \phi^i - \sum_{i<j} [\phi^i, \phi^j][\phi^i, \phi^j] + \bar{\Psi} \Gamma^\mu D_\mu \Psi + \bar{\Psi} \Gamma^i [\phi^i, \Psi] \right) \tag{2.6}$$

All fields are $MN \times MN$ matrices. With the notation

$$\mathbf{A} = \frac{1}{\sqrt{2}} (\phi^1 + i\phi^2)\,, \qquad \mathbf{B} = \frac{1}{\sqrt{2}} (\phi^3 + i\phi^4)\,, \qquad \mathbf{\Phi} = \frac{1}{\sqrt{2}} (\phi^5 + i\phi^6)\,. \tag{2.7}$$

the elements of the bosonic fields which survive the projection (2.2) are

$$\mathbf{\Phi} \equiv \begin{pmatrix} \Phi_1 & 0 & \cdots & 0 \\ 0 & \Phi_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \Phi_M \end{pmatrix} \qquad A_\mu \equiv \begin{pmatrix} A_{\mu 1} & 0 & \cdots & 0 \\ 0 & A_{\mu 2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & A_{\mu M} \end{pmatrix} \tag{2.8}$$

JHEP11(2004)053

and

$$\mathbf{A} \equiv \begin{pmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{M-1} \\ A_M & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \mathbf{B} \equiv \begin{pmatrix} 0 & 0 & \cdots & 0 & B_M \\ B_1 & 0 & \cdots & 0 & 0 \\ 0 & B_2 & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{M-1} & 0 \end{pmatrix}. \tag{2.9}$$

Each non-vanishing entry of the above matrices is an $N \times N$ matrix and corresponds to a bosonic field of the $\mathcal{N} = 2$ theory.

The $\mathcal{N} = 4$ spinor Ψ contains four different complex Weyl spinors $\chi_{\mathbf{A}}$,$\chi_{\mathbf{B}}$,$\psi_{\mathbf{\Phi}}$ and ψ so that, with the definition in (2.7) $\chi_{\mathbf{A}}$, $\chi_{\mathbf{B}}$ and $\psi_{\mathbf{\Phi}}$ are the superpartners of $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{\Phi}$, respectively, and ψ is the gaugino. Then

$$\psi_{\mathbf{\Phi}} \equiv \begin{pmatrix} \psi_{\Phi 1} & 0 & \cdots & 0 \\ 0 & \psi_{\Phi 2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_{\Phi M} \end{pmatrix} \qquad \psi \equiv \begin{pmatrix} \psi_1 & 0 & \cdots & 0 \\ 0 & \psi_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \psi_M \end{pmatrix} \tag{2.10}$$

and

$$\chi_{\mathbf{A}} \equiv \begin{pmatrix} 0 & \chi_{A1} & 0 & \cdots & 0 \\ 0 & 0 & \chi_{A2} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \chi_{AM-1} \\ \chi_{AM} & 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \chi_{\mathbf{B}} \equiv \begin{pmatrix} 0 & 0 & \cdots & 0 & \chi_{BM} \\ \chi_{B1} & 0 & \cdots & 0 & 0 \\ 0 & \chi_{B2} & \cdots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \chi_{BM-1} & 0 \end{pmatrix}. \tag{2.11}$$

An element of the residual gauge group is

$$U \equiv \begin{pmatrix} U^{(1)} & 0 & \cdots & 0 \\ 0 & U^{(2)} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & U^{(M)} \end{pmatrix} \tag{2.12}$$

and acts on all of the above matrices by conjugation, $M \to UMU^\dagger$ (with appropriate additional terms for the gauge field).

The action for scalar fields is

$$\mathcal{L}_{\text{scalar}} = \frac{1}{g_{\text{YM}}^2 M} \sum_{I=1}^{M} \text{TR}\left(D_\mu \mathbf{A}_I D_\mu \bar{\mathbf{A}}_I + D_\mu \mathbf{B}_I D_\mu \bar{\mathbf{B}}_I + \frac{1}{2} D_\mu \mathbf{\Phi}_I D_\mu \bar{\mathbf{\Phi}}_I \right) + \mathcal{L}_D + \mathcal{L}_F\,, \tag{2.13}$$

where the interaction F- and D-terms can be gotten from

$$\mathcal{L}_D = -\tfrac{1}{g_{\text{YM}}^2} \text{TR}\left([\mathbf{A}, \bar{\mathbf{A}}] + [\mathbf{B}, \bar{\mathbf{B}}] + [\mathbf{\Phi}, \bar{\mathbf{\Phi}}] \right)^2, \tag{2.14}$$

$$\mathcal{L}_F = \tfrac{2}{g_{\text{YM}}^2} \text{TR}\left(|[\mathbf{A}, \mathbf{B}]|^2 + |[\mathbf{A}, \mathbf{\Phi}]|^2 + |[\mathbf{B}, \mathbf{\Phi}]|^2 \right). \tag{2.15}$$

JHEP11(2004)053

2.2 IIB string on $AdS_5 \times \mathbf{S}^5/\mathbf{Z}_M$

The $\mathcal{N}=2$ theory is the holographic dual of IIB string theory with background the orbifold $AdS_5 \times \mathbf{S}^5/\mathbf{Z}_M$ and with MN units of Ramond-Ramond 5-form flux through the 5-sphere. Since the 5-sphere contains M copies of a fundamental domain that are identified by the orbifold group, there are N units of flux per fundamental domain. The action of the orbifold group is obtained by embedding the 5-sphere in $\mathbf{R}^6 \sim \mathbf{C}^3$ so that

$$\sum_{i=1}^{3} |z_i|^2 = R^2$$

where $(z_1, z_2, z_3) \in \mathbf{C}^3$ and then identifying points as prescribed in (2.1):

$$\big(z_1, z_2, z_3\big) \sim \big(\omega z_1, \omega^{-1} z_2, z_3\big)\,, \qquad \omega = e^{2\pi i/M} \tag{2.16}$$

The radii of AdS_5 and $\mathbf{S}^5$ are equal and are given by

$$R^2 = \sqrt{4\pi g_s \alpha'^2 NM}\,, \tag{2.17}$$

where g_s is the type-IIB string coupling. Furthermore, the Yang-Mills theory coupling constant of the parent $\mathcal{N}=4$ theory is identified with the coupling constant of the parent superstring theory on $AdS_5 \times \mathbf{S}^5$,

$$4\pi g_s = g_{YM}^2\,. \tag{2.18}$$

2.3 Double scaling limit

We shall consider the double scaling limit of both the gauge theory and its string theory dual. The double scaling limit of the string theory is the Penrose limit which obtains the pp-wave background. The radii of AdS_5 and $\mathbf{S}^5$, given by R in (2.17), are put to infinity by scaling both N and M to infinity while keeping g_s small but finite. The parameter which will become the null compactification radius, $R^- = R^2/2M$, is also held fixed in the limit by keeping the ratio N/M fixed.

The metric of $AdS_5 \times \mathbf{S}^5/\mathbf{Z}_M$ can be written as:

$$\begin{aligned} ds^2 \,=\, R^2 \Big[& -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2 + \\ & d\alpha^2 + \sin^2\alpha\, d\theta^2 + \cos^2\alpha\, \big(d\gamma^2 + \cos^2\gamma\, d\chi^2 + \sin^2\gamma\, d\phi^2\big)\Big]\,. \end{aligned} \tag{2.19}$$

The angles of $\mathbf{S}^5$ are related to the complex coordinates of $\mathbf{C}^3/\mathbf{Z}_M$ by

$$z_1 = R\cos\alpha\cos\gamma\, e^{i\chi}, \quad z_2 = R\cos\alpha\sin\gamma\, e^{i\phi}\,, \quad z_3 = R\sin\alpha e^{i\theta} \tag{2.20}$$

In terms of the angles of $\mathbf{S}^5$ the orbifold described by the action (2.16) is obtained by the identifications

$$\chi \sim \chi + \frac{2\pi}{M}\,, \qquad \phi \sim \phi - \frac{2\pi}{M}\,. \tag{2.21}$$

To take the Penrose limit it is useful to introduce the coordinates

$$r = \rho R\,, \qquad w = \alpha R\,, \qquad y = \gamma R\,. \tag{2.22}$$

and the light-cone coordinates

$$x^+ = \frac{1}{2}\,(t+\chi)\,, \qquad x^- = \frac{R^2}{2}\,(t-\chi)\,. \tag{2.23}$$

After taking the $R \to \infty$ limit and renaming some coordinates, the metric becomes [38]

$$ds^2 = -4dx^+dx^- - \sum_{i=1}^{8}(x^i)^2\,{dx^+}^2 + \sum_{i=1}^{8}{dx^i}^2\,, \tag{2.24}$$

In the geometry (2.24) there is also a Ramond-Ramond flux

$$F_{+1234} = F_{+5678} = \text{const.} \tag{2.25}$$

So far, with the rescaling (2.22) and (2.23) the only limit that we have taken to obtain (2.24) is that of large R. The orbifold identification (2.21) implies that the light-cone coordinates have the periodicity

$$\begin{aligned} x^+ &\sim x^+ + \frac{\pi}{M} \\ x^- &\sim x^- + \frac{\pi R^2}{M}\,, \end{aligned} \tag{2.26}$$

In the double scaling limit, as R is taken large, M is also taken large so that $R^- = \frac{R^2}{2M}$ is held fixed. In the limit

$$\left(x^+, x^-\right) \sim \left(x^+, x^- + 2\pi R^-\right). \tag{2.27}$$

The periodic direction becomes null. As a consequence the corresponding light-cone momentum $2p^+$ is quantized in units of $1/R^-$.

The conclusion is that the Penrose limit of $AdS_5 \times \mathbf{S}^5/\mathbf{Z}_M$ with $M \to \infty$ in this particular way leads to a Discrete Light-Cone Quantization (DLCQ) of the string on a pp-wave background, in which the null coordinate x^- is periodic. Note that the orbifold of the 5-sphere preserves half of the supersymmetries of the original $AdS_5 \times \mathbf{S}^5$ solution of string theory. Nonetheless, in the Penrose limit, we recover the maximally supersymmetric plane-wave background.

Discrete light-cone quantization of the string on the pp-wave background is a slight generalization of ref. [13]. One component of the light-cone momentum is quantized as

$$2p^+ = \frac{k}{R^-}\,, \qquad k = 1,2,3,\ldots \tag{2.28}$$

The other component is the light-cone-gauge hamiltonian,

$$\begin{aligned} 2p^- &= \sum_{n=-\infty}^{\infty}\left(\sum_{i=1}^{8} a_n^{i\dagger}a_n^i + \sum_{\alpha=1}^{8} b_n^{\alpha\dagger}b_n^\alpha\right)\sqrt{1+\frac{4n^2(R^-)^2}{k^2\alpha'^2}} \\ &= \sum_{n=-\infty}^{\infty}\left(\sum_{i=1}^{8} a_n^{i\dagger}a_n^i + \sum_{\alpha=1}^{8} b_n^{\alpha\dagger}b_n^\alpha\right)\sqrt{1+\frac{4\pi g_s N}{M}\frac{n^2}{k^2}} \end{aligned} \tag{2.29}$$

where $a_n^i, a_n^{i\dagger}$ and $b_n^\alpha, b_n^{\alpha\dagger}$ are the annihilation and creation operators for the discrete bosonic and fermionic transverse oscillations of the string, respectively. They obey the (anti-) commutation relation

$$\left[a_{n_1}^i, a_{n_j}^{j\dagger}\right] = \delta^{ij}\delta_{n_i n_j}\,, \qquad \left\{b_{n_1}^\alpha, b_{n_j}^{\beta\dagger}\right\} = \delta^{\alpha\beta}\delta_{n_i n_j}\,. \tag{2.30}$$

In the last line of eq. (2.29) we have written the compactification radius in terms of string background parameters.

There are also wrapped states. If the total number of times that the closed string wraps the compact null direction is m, the level-matching condition is

$$km = \sum_{n=-\infty}^{\infty} n\left(\sum_{i=1}^{8} a_n^{i\dagger}a_n^i + \sum_{\alpha=1}^{8} b_n^{\alpha\dagger}b_n^\alpha\right), \tag{2.31}$$

States of the string are characterized by their discrete light-cone momentum k and their wrapping number m. The lowest energy state in a given sector is the string sigma model vacuum, $|k,m\rangle$ which obeys

$$a_n^i\,|k,m\rangle = 0 = b_n^\alpha\,|k,m\rangle\,, \qquad \forall n, i, \alpha\,.$$

Other string states are built from the vacuum by acting with transverse oscillators,

$$\prod_{j=1}^{L} a_{n_j}^{i_j\,\dagger} \prod_{j'=1}^{L'} b_{n_{j'}}^{\alpha_{j'}\,\dagger}\mid k,m\rangle\,. \tag{2.32}$$

The level matching condition reads

$$\sum_{j=1}^{L} n_j + \sum_{j'=1}^{L'} n_{j'} = k\,m\,. \tag{2.33}$$

2.4 Matching charges

There are three important quantum numbers that can be matched between the string theory and its gauge theory dual. One is the energy in string theory, which is the quantum operator generating a flow along the Killing vector field $i\partial_t$ of the background. It corresponds to the conformal dimension, Δ, of operators in the gauge theory.

The others are U(1) charges. Two are particularly important to us. One is J' which generates a U(1) which is in the SU(2) subgroup of the R-symmetry

$$A \to e^{i\xi}A\,, \qquad B \to e^{i\xi}B\,, \qquad 0 \le \xi < 2\pi$$

J' which has integer eigenvalues. In the orbifold geometry, it corresponds to the Killing vector $J' = -\frac{i}{2}\left(\partial_\chi + \partial_\phi\right)$. There is an additional U(1) which is not part of the R-symmetry

$$A \to e^{i\zeta}A\,, \qquad B \to e^{-i\zeta}B\,, \qquad 0 \le \zeta < \frac{2\pi}{M}\,.$$

	Δ	MJ	J'	$2p^-$
A_I	1	1/2	1/2	0
B_I	1	$-1/2$	1/2	1
Φ_I	1	0	0	1
χ_{AI}	3/2	1/2	0	1
χ_{BI}	3/2	$-1/2$	0	2
$\psi_{\Phi I}$	3/2	0	$-1/2$	2
ψ_I	3/2	0	$-1/2$	2

Table 1: Dimensions and charges for chiral fields and gauginos.

	Δ	MJ	J'	$2p^-$
$\overline{A}_I$	1	$-1/2$	$-1/2$	2
$\overline{B}_I$	1	1/2	$-1/2$	1
$\overline{\Phi}_I$	1	0	0	1
$\overline{\chi}_{AI}$	3/2	$-1/2$	0	2
$\overline{\chi}_{BI}$	3/2	1/2	0	1
$\overline{\psi}_{\Phi I}$	3/2	0	1/2	1
$\overline{\psi}_I$	3/2	0	1/2	1

Table 2: Dimensions and charges for complex conjugate fields.

The domain of the angle ζ is reduced from 2π to $2\pi/M$ by the orbifold identification. This U(1) is generated by J whose eigenvalues are integer multiples of M. In order to normalize it more conveniently, we rename it MJ where J has integer eigenvalues. On the orbifold geometry, it corresponds to the Killing vector $J = -\frac{i}{2M}(\partial_\chi - \partial_\phi)$.

In summary, charges and Killing vectors are related by

$$\Delta = i\partial_t\,, \qquad J = -\frac{i}{2M}(\partial_\chi - \partial_\phi)\,, \qquad J' = -\frac{i}{2}(\partial_\chi + \partial_\phi)\,.$$

We can then recall the combinations of χ, ϕ and t which were used to form the light-cone coordinates x^+ and x^- of the pp-wave geometry to deduce the light-cone momenta

$$\begin{aligned} 2p^- &= i(\partial_t + \partial_\chi) = \Delta - MJ - J' \\ 2p^+ &= i\frac{(\partial_t - \partial_\chi)}{R^2} = \frac{\Delta + MJ + J'}{2MR^-}\,. \end{aligned} \tag{2.34}$$

These are the light-cone momenta of string states. We will focus on those states of the gauge theory where these quantum numbers remain finite in the double scaling limit. It will be easy to see that $2p^+$ will turn out to be quantized appropriately in units of integers$/2R^-$ and the values of $2p^-$ which we find in the gauge theory will be compared to the spectrum of the string light-cone hamiltonian.

The BPS condition $\Delta \geq |MJ + J'|$ implies that keeping $2p^+$ and $2p^-$ finite as $R, M \to \infty$ will clearly only be possible when both Δ and $MJ + J'$ diverge with their difference, $\Delta - (MJ + J')$, remaining finite.

The charges of gauge theory operators are obtained as follows. By convention, the U(1) transformation is generated by $e^{4\pi i J}$ so the A_I and B_I fields that make up the hypermultiplets have fractional charge under J, $1/2M$ and $-1/2M$ respectively. The operator J' generates a U(1) symmetry contained in the $\mathrm{SU}(2)_R$ factor of the R-symmetry. Under this $\mathrm{U}(1) \subset \mathrm{SU}(2)_R$, the fields Φ_I are neutral. On the other hand, the scalars A_I, B_I in the hypermultiplets both have charge 1/2 under J'. Complex conjugation and supersymmetry give the remaining charge assignments, for the fermions and all the conjugate fields.

The dimension and charge assignments, along with the $2p^-$ values, are summarized in tables 1 and 2. In table 1, A_I, B_I refer to the scalar components of the $\mathcal{N} = 1$ chiral superfields that form the $\mathcal{N} = 2$ hypermultiplets. χ_{A_I}, χ_{B_I} are their fermionic partners.

JHEP11(2004)053

Φ_I are the complex scalars in the vector multiplet, while ψ_{Φ_I} are their fermionic partners. Finally, ψ_I are the gauginos in the theory. Table 2 lists the complex conjugate fields.

2.5 The holographic dictionary

In order to identify states in the $\mathcal{N} = 2$ gauge theory with finite values of light-cone momenta, as given in (2.34), we first find the appropriate quantum numbers of the field operators. These are tabulated in table 1 and table 2. We see that only the fields A_I carry vanishing $2p^-$. By matching quantum numbers, we see that the string state $|k, 0>$ corresponds to the gauge invariant composite operator

$$|k,0\rangle \quad \leftrightarrow \quad \mathrm{TR}\left((A_1(x)A_2(x)\ldots A_M(x))^k\right).$$

We have indicated the x-dependence of the composite operator. In the following, where from the context it is obvious, we will omit it. Because $A_I(x)$ transforms in the bi-fundamental representation of the gauge group, we are required to form the chains $A_1 \ldots A_M$ to obtain a gauge invariant operator. This chain can be repeated k times. The conformal dimension of this composite operator is protected by supersymmetry. This protection is inherited from the parent $\mathcal{N} = 4$ theory. Thus, its exact conformal dimension is $\Delta = kM$ and its exact spectrum is therefore $p^- = 0$.

We have chosen to use a one-trace operator to represent the single string state. Indeed, this choice has some arbitrariness. A more general operator would be any linear combination of multi-trace operators,

$$\mathcal{O}(\ell_1, \ell_2, \ldots) = (\mathrm{TR}\,(A_1 \ldots A_M))^{\ell_1} \left(\mathrm{TR}\,(A_1 \ldots A_M)^2\right)^{\ell_2} \left(\mathrm{TR}\,(A_1 \ldots A_M)^3\right)^{\ell_3} \ldots \tag{2.35}$$

where

$$\sum_i \ell_i = k\,.$$

Operators with different trace structures are not mixed in the planar limit of Yang-Mills theory, but non-planar corrections do mix them, though the mixing vanishes in the double scaling limit. One natural way to decide which amongst the degenerate states are relevant is to diagonalize the inner product $\langle \ell_1, \ldots | \ell_1', \ldots \rangle$ which one would obtain from the correlation function

$$\left\langle \bar{\mathcal{O}}(x; \ell_1, \ell_2, \ldots)\mathcal{O}(y; \ell_1', \ell_2', \ldots)\right\rangle = \frac{1}{(x-y)^{2k}} \left\langle \ell_1, \ldots | \ell_1', \ldots \right\rangle.$$

However, there is no natural way to decide which of the resulting states is a one-string state, two-string state, etc. This is similar to the problem on the string side of trying to distinguish the multi-string states

$$|\ell_1, 0\rangle \otimes |\ell_2, 0\rangle \otimes |\ell_3, 0\rangle \otimes \cdots$$

which, when $\sum \ell_i = k$, all have the same quantum numbers. At this point, this should be regarded as an open problem. Fortunately, we shall find that we do not have to solve this problem here since we are interested in the eigenvalues of the hamiltonian which are independent of the basis. We already know that these states are degenerate and have eigenvalue $2p^- = 0$.

JHEP11(2004)053

There are eight states which are created by one bosonic oscillator and eight which are created by a fermionic oscillator. These all add one unit to the hamiltonian, $2p^-$. In Yang-Mills theory, they are gotten by inserting an impurity into the $A_1 \ldots A_M$ chains. From tables 1 and 2, we see that four of the bosonic states are gotten by inserting B_I, Φ_I, $\bar{B}_I$ or $\bar{\Phi}_I$. The other four are gotten by replacing A_I by a derivative of A_I. For example, a state with $2p^- = 1$+corrections is

$$\left(a_n^{5\dagger} + ia_n^{6\dagger}\right)|k,m\rangle \quad \leftrightarrow \quad \sum_{I=1}^{kM} e^{2\pi inI/kM}\mathrm{TR}\left(A_1 \ldots A_{I-1}\Phi_I A_I \ldots A_M \left(A_1 \ldots A_M\right)^k\right). \tag{2.36}$$

We have superposed over positions at which the impurity could be inserted. The momentum in the insertion n coincides with the world-sheet momentum of the oscillator state. The level matching condition comes from realizing that the actual periodicity of the operator is $I \to I + M$, rather than $I \to I + kM$, which the plane waves anticipate. This requires that $n = km$, where m is an integer. This is the level matching condition. The integer m is identified with the wrapping number of the world-sheet on the compact coordinate.

It is worth stressing that the single impurity operators do not appear in the $\mathcal{N} = 4$ case, as they do not satisfy the level matching condition. Here instead, the single oscillator state in (2.36) is no longer a protected operator. Its dimension Δ should get radiative corrections beyond the tree level in Yang-Mills theory, even for planar diagrams. In fact, it must get such corrections if it is to match the string spectrum,

$$2p^- = \sqrt{1 + \frac{g_{YM}^2 N}{M}\frac{n^2}{k^2}} \tag{2.37}$$

for planar diagrams. Indeed, we shall see in the following that it produces this spectrum to one order in g_{YM}^2. We will also learn that this operator is quasi-protected in that, in the double scaling limit, all non-planar corrections to (2.37) vanish. *Our Yang-Mills computation predicts that the spectrum of this state in string theory does not receive string loop corrections.*

Here, one might wonder why, rather than (2.36) we couldn't insert one impurity into a multi-trace operator. Indeed, when $k > 1$ there are multi-trace operators which have the same k and m and which non-planar diagrams mix with (2.36). Moreover, since this mixing vanishes in the double scaling limit, all such operators with the same m and k are exactly degenerate. Again, one could choose a special basis by diagonalizing their inner product, but we shall not have to do this, as here we are interested only in questions about the spectrum which are basis independent.

The winding state with two impurities which corresponds to two oscillator states and have energies $2p^- = 2$+corrections reads

$$\begin{aligned} &\left(a_{n_1}^{5\dagger} + ia_{n_1}^{6\dagger}\right)\left(a_{n_2}^{5\dagger} + ia_{n_2}^{6\dagger}\right)|k,m\rangle \quad \leftrightarrow \\ &\leftrightarrow \quad \sum_{I,J=1}^{kM} \mathrm{TR}\left[A_1 \ldots A_{I-1}\Phi_I A_I \ldots A_{J-1}\Phi_J A_J \ldots A_M \left(A_1 \ldots A_M\right)^{k-1}\right] e^{2\pi i \frac{In_1 + Jn_2}{Mk}}. \end{aligned} \tag{2.38}$$

JHEP11(2004)053

Here, the world-sheet momenta are n_1 and n_2. Also, note that the state is periodic under translating both I and J by M. This leads to the quantization condition $n_1 + n_2 = km$. We interpret this as the level matching condition and m is the winding number.

In most of the above discussion, we have focused on the oscillators constructed out of Φ. However, it is straightforward to see that similar expressions hold for the remaining oscillators, with Φ suitably replaced by the other type of impurities or one of the fermionic fields.

3. Spectrum of strings from Yang-Mills theory

It is by now well known that composite operators made from gauge invariant products of adjoint (and in our case bi-fundamental) fields, have some special properties. For example, consider a composite made from the scalar fields of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory,

$$\mathrm{TR}\left(\phi^{i_1}\phi^{i_2}\ldots\phi^{i_k}\right).$$

In four space-time dimensions, the engineering dimension of a scalar field is one and therefore, for all choices of the indices, $i_1, \ldots, i_k$, the composite operators above have the same tree level dimension.

Of course, at the quantum level, when radiative corrections are taken into account, the degeneracy between such operators can be lifted. Generally, loop corrections can be separated into two different kinds. One type are "flavor blind", they do not distinguish between the different flavors, labelled by indices $i_1, \ldots, i_k$, but only proceed through the fact that the fields carry charges which couple to the gauge field. These corrections provide the same overall constant shift in the spectrum of all of the operators and do not resolve the degeneracy between them. In theories with enough supersymmetry, these corrections can cancel identically. This is indeed the case in the parent $\mathcal{N} = 4$ theory and in the $\mathcal{N} = 2$ theory of interest. An example of interactions which contribute to these radiative corrections are the scalar four-point couplings in the D-terms in (2.14). When combined with gauge field loops, the interactions from D-terms cancel identically.

The other kind of radiative corrections do distinguish between different flavors. They can split the tree level degeneracy of conformal dimensions. A well-known example occurs in the $\mathcal{N} = 4$ gauge theory where the F-terms couple the tree level degenerate operators and act effectively like the integrable hamiltonian for an SO(6) spin chain [39]. Again, in the example of interest to us, the $\mathcal{N} = 2$ theory, the F-terms (2.15) in the scalar four-point couplings also split the spectrum of conformal dimensions. It can be shown that they account for the entire radiative correction to one loop order for the conformal dimensions of products of scalar fields that we shall consider here. The contribution is analyzed in appendix A.

Just as in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [40, 41], the computation of anomalous dimensions is elegantly summarized by the action of an effective hamiltonian. In appendix A we show that the one-loop shift in dimension of all composite operators which have a certain property can be summarized by the action on traces of matrices of the dilatation operator

$$D = \sum_{I=1}^{M} \mathrm{TR}(A_I\bar{A}_I + B_I\bar{B}_I + \Phi_I\bar{\Phi}_I) - \tag{3.1}$$
$$-\frac{g_{YM}^2 M}{8\pi^2}\sum_{I=1}^{M}\mathrm{TR}\Big[A_I\Phi_{I+1}\bar{A}_I\bar{\Phi}_I - A_I\Phi_{I+1}\bar{\Phi}_{I+1}\bar{A}_I - \Phi_I A_I\bar{A}_I\bar{\Phi}_I + \Phi_I A_I\bar{\Phi}_{I+1}\bar{A}_I\Big] + \cdots .$$

The terms with dots contain other matrices such as B_I and $\bar{B}_I$ and fermions which we will not use here. The operatorial property is defined by the Wick contractions

$$\big\langle[\bar{A}_I]_{ab}[A_J]_{cd}\big\rangle_0 = \delta_{IJ}\delta_{ad}\delta_{bc}\,, \quad \big\langle[\bar{B}_I]_{ab}[B_J]_{cd}\big\rangle_0 = \delta_{IJ}\delta_{ad}\delta_{bc}\,, \quad \big\langle[\bar{\Phi}_I]_{ab}[\Phi_J]_{cd}\big\rangle_0 = \delta_{IJ}\delta_{ad}\delta_{bc}\,.$$

An example of a basis of operators (with one impurity and $k = 2$) is the set of $2M$ elements

$$\mathrm{TR}(A_1\dots A_{I-1}\Phi_I A_I\dots A_M A_1\dots A_M)\,, \qquad \mathrm{TR}(A_1\dots A_{I-1}\Phi_I A_I\dots A_M)\mathrm{TR}(A_1\dots A_M)$$

for each value of $I = 1,\dots,M$. To operate on a such basis with D we Wick-contract $\bar{A}_I$ and $\bar{\Phi}_I$ which occur in D with each of the A_I and Φ_I which occur in the operators, respectively. In each case this produces a linear combination of operators in the basis. The first term in (3.1) gives the tree level contribution to the conformal dimension and the second term gives the one loop correction.

Generally, there is a basis of operators $\mathcal{O}_\alpha$ determined by the quantum number k and the number and types of impurities and

$$D\mathcal{O}_\alpha = D_\alpha{}^\beta\mathcal{O}_\beta\,.$$

The eigenvectors of the matrix $D_\alpha{}^\beta$ are the scaling operators and the eigenvalues are the conformal dimensions.

One immediate result is that the set of operators of the type (2.35), since they contain only A_I's, do not get corrections at all. They are related to chiral primary operators of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory and share that property. For them, D is diagonal and its eigenvalue is kM.

In the next sections we will diagonalize the matrix $D_\alpha{}^\beta$ for operators with one impurity and any value of the discrete light cone momentum (which, as we shall show, do not have an anomalous dimension beyond the planar level) and for operators with two impurities and the first few values of the discrete light cone momentum $k = 1, 2, 3$. Except for $k = 1$ the eigenstates of the dilatation matrix will not be eigenstates of the winding number m but will be linear combinations of these eigenstates with different values of m.

The string spectrum can be obtained exactly in the large M limit where the variable $x = I/M$ becomes continuous and the action of the hamiltonian on states is described by a simple differential operator whose eigenstates and eigenvalues can be easily found. We shall also make some comments on possible extensions of our results to the case of higher values of k and a larger number of impurities.

4. One impurity = one oscillator states

Let us begin by considering the simplest states, those which have $k = 1$. With no impurities, we already know that this state has $2p^- = 0$. With one impurity it is convenient to take the linear combination

$$\mathcal{O}_{n,1} = \sum_{I=1}^{M} O_I^1 \omega^{nI} \tag{4.1}$$

where

$$O_I^1 = \mathrm{TR}(A_1 A_2 \cdots A_{I-1} \Phi_I A_I \cdots A_M)\,. \tag{4.2}$$

Here $\omega = e^{\frac{2\pi i}{M}}$ and since $k = 1$, the level matching condition, $n = mk = m$, is trivially obeyed when we identify m as both the world-sheet momentum and the wrapping number. (4.2) is to be treated as periodic in I with period M. This has been anticipated in the Fourier transform (4.1).

The action of the dilatation operator (3.1) on (4.1) is found by performing Wick contractions.

$$\begin{aligned} D \circ \mathcal{O}_{n,1} &= (M+1)\mathcal{O}_{n,1} - \frac{g_{YM}^2 MN}{8\pi^2} \sum_{I=1}^{M} \left(O_{I+1}^1 - 2O_I^1 + O_{I-1}^1\right)\omega^{nI} \\ &= \left[M + 1 + \frac{g_{YM}^2 MN}{8\pi^2} 2\left(1 - \cos\frac{2\pi n}{M}\right)\right] \mathcal{O}_{n,1}\,. \end{aligned} \tag{4.3}$$

In the large M limit we can expand the cosine up to $\mathcal{O}(1/M^2)$. We also recall that, in this case $2p^- = D - M$ to get

$$2p^- \circ \mathcal{O}_{n,1} = \left[1 + \frac{1}{2}\frac{g_{\mathrm{YM}}^2 N}{M} n^2\right] \mathcal{O}_{n,1}\,. \tag{4.4}$$

The string theory result is given in (2.37), expanding it in powers of $g_s \frac{N}{M} = \frac{g_{YM}^2 N}{4\pi M}$ one finds that (4.4) provides the exact one loop correction. This confirms the proposed operator-state map to this order in expansion in the world-sheet momentum, n.

Now, consider the case where $k = 2$. We find that the two states

$$\begin{aligned} O_+ = \sum_{I=1}^{M} e^{2\pi i m I/M} \Big\{ &\mathrm{TR}(A_1 \ldots \Phi_I \ldots A_M A_1 \ldots A_M) + \\ &+ \mathrm{TR}(A_1 \ldots \Phi_I \ldots A_M)\mathrm{TR}(A_1 \ldots A_M) \Big\} \end{aligned} \tag{4.5}$$

$$\begin{aligned} O_- = \sum_{I=1}^{M} e^{2\pi i m I/M} \Big\{ &\mathrm{TR}(A_1 \ldots \Phi_I \ldots A_M A_1 \ldots A_M) - \\ &- \mathrm{TR}(A_1 \ldots \Phi_I \ldots A_M)\mathrm{TR}(A_1 \ldots A_M) \Big\} \end{aligned} \tag{4.6}$$

are exact eigenstates of the D with eigenvalues

$$D \circ O_+ = \left[2M + 1 + \frac{g_{YM}^2 M(N+1)}{4\pi^2}\left(1 - \cos\frac{2\pi m}{M}\right)\right] O_+ \tag{4.7}$$

$$D \circ O_- = \left[2M + 1 + \frac{g_{YM}^2 M(N-1)}{4\pi^2}\left(1 - \cos\frac{2\pi m}{M}\right)\right] O_- \tag{4.8}$$

respectively.

These eigenstates of the dilatation operator have an interesting form. They are a mixture of one-trace and two-trace operators which we would normally associate with one-string and two-string states and the mixing does not depend on the coupling constant, so does not go away when the coupling constant is made small.

Further, note that, in the double scaling limit, these eigenvalues are degenerate. This means that string loop corrections vanish in this limit and the free string spectrum is exact.

What about the case $k = 3$? In this case there are a number of degenerate states

$$\mathcal{O}_1 = \sum_{I=1}^{M} e^{2\pi i m I/M} \mathrm{TR}\left(A_1 \ldots \Phi_I \ldots A_M (A_1 \ldots A_M)^2\right) \tag{4.9}$$

$$\mathcal{O}_2 = \sum_{I=1}^{M} e^{2\pi i m I/M} \mathrm{TR}\left(A_1 \ldots \Phi_I \ldots A_M (A_1 \ldots A_M)\right) \mathrm{TR}\left(A_1 \ldots A_M\right) \tag{4.10}$$

$$\mathcal{O}_3 = \sum_{I=1}^{M} e^{2\pi i m I/M} \mathrm{TR}\left(A_1 \ldots \Phi_I \ldots A_M\right) \mathrm{TR}\left((A_1 \ldots A_M)^2\right) \tag{4.11}$$

$$\mathcal{O}_4 = \sum_{I=1}^{M} e^{2\pi i m I/M} \mathrm{TR}\left(A_1 \ldots \Phi_I \ldots A_M\right) \mathrm{TR}\left(A_1 \ldots A_M\right) \mathrm{TR}(A_1 \ldots A_M)\,. \tag{4.12}$$

All operators have the same tree level dimension $3M + 1$. Here, $k = 3$ and m is the total wrapping number of each state. Note that the world-sheet momentum in $\mathcal{O}_1$ is $n = m/3$, in $\mathcal{O}_2$ is $n = m/2$ and in $\mathcal{O}_3$ and $\mathcal{O}_4$ it is $n = m$. It satisfies the level matching condition $n = km$ for single string states in each case if we interpret a single trace as a single trace state. Again, eigenvectors of the dilatation operator are found by choosing simple linear combinations,

$$\begin{aligned}
D \circ (\mathcal{O}_1 + \mathcal{O}_2 - \mathcal{O}_3 - \mathcal{O}_4) &= \left[3M + 1 + \frac{g_{YM}^2 M(N-1)}{4\pi^2}\left(1 - \cos\frac{2\pi m}{M}\right)\right] \times \\
&\quad \times (\mathcal{O}_1 + \mathcal{O}_2 - \mathcal{O}_3 - \mathcal{O}_4) \\
D \circ (2\mathcal{O}_1 + 2\mathcal{O}_2 + \mathcal{O}_3 + \mathcal{O}_4) &= \left[3M + 1 + \frac{g_{YM}^2 M(N+2)}{4\pi^2}\left(1 - \cos\frac{2\pi m}{M}\right)\right] \times \\
&\quad \times (2\mathcal{O}_1 + 2\mathcal{O}_2 + \mathcal{O}_3 + \mathcal{O}_4) \\
D \circ (\mathcal{O}_1 - \mathcal{O}_2 + \mathcal{O}_3 - \mathcal{O}_4) &= \left[3M + 1 + \frac{g_{YM}^2 M(N+1)}{4\pi^2}\left(1 - \cos\frac{2\pi m}{M}\right)\right] \times \\
&\quad \times (\mathcal{O}_1 - \mathcal{O}_2 + \mathcal{O}_3 - \mathcal{O}_4) \\
D \circ (2\mathcal{O}_1 - 2\mathcal{O}_2 - \mathcal{O}_3 + \mathcal{O}_4) &= \left[3M + 1 + \frac{g_{YM}^2 M(N-2)}{4\pi^2}\left(1 - \cos\frac{2\pi m}{M}\right)\right] \times \\
&\quad \times (2\mathcal{O}_1 - 2\mathcal{O}_2 - \mathcal{O}_3 + \mathcal{O}_4)
\end{aligned}$$

Again, we see that the degeneracy is split, but by terms which will vanish in the double scaling limit. In that limit, the four states are degenerate again.

The situation will be similar with any value of k. Generally, the dilatation operator mixes operators with different distributions of traces. However, in the case of a single impurity, is straightforward to show that this mixing vanishes in the double scaling limit. This gives the one-impurity state an interesting property, they get Yang-Mills loop corrections to all orders in perturbation theory from planar diagrams, but all non-planar diagrams vanish. This is interpreted in the string theory as the absence of string loop corrections to the free string spectrum.

Let us notice that our explicit gauge theory computations confirme that $g^2_{\rm Y.M.}\frac{N}{M}$ and $g_2 = \frac{M}{N}$ are respectively the effective 't Hooft coupling and the (two dimensional) string coupling. It is useful to mention that the above result is indeed what one expects from the usual ($\mathcal{N}=4$) BMN computations. Namely, if in the expression for $\lambda' = g^2_{\rm Y.M.}\frac{N}{J^2}$ we replace N by NM and J with M, we obtain $\lambda' = g^2_{\rm Y.M.}\frac{N}{M}$ and similarly $\frac{J^2}{N}$ becomes $\frac{M}{N}$.

5. Two impurities, $k = 1$

Let us now consider the string state with two oscillators and one-unit of light cone momentum $k = 1$. The gauge theory operators dual to this state can be obtained for example by a double insertion of Φ fields into a string of A fields

$$O^1_{IJ} = \mathrm{TR}(A_1 A_2 \cdots A_{I-1}\Phi_I A_I \cdots A_{J-1}\Phi_J A_J \cdots A_M)\,. \tag{5.1}$$

This is a set of $M(M+1)/2$ independent operators ($I \le J$). At tree level, they are degenerate, with conformal dimension $M + 2$.

The action of the dilatation operator (3.1) on the states (5.1) is given by

$$(D - 2M - 2)\circ O^1_{IJ} = \frac{g^2_{YM} MN}{8\pi^2}\left(-\left(\nabla^2_I + \nabla^2_J\right) O^1_{IJ} - \delta_{IJ}\left(\nabla_J - \hat{\nabla}_I\right) O^1_{IJ}\right) \tag{5.2}$$

where we have introduced the forward and backwords shift operators defined by

$$\begin{aligned} \nabla_I O^1_{IJ} &= O^1_{I+1,J} - O^1_{IJ} \\ \hat{\nabla}_I O^1_{IJ} &= O^1_{IJ} - O^1_{I-1,J} \end{aligned} \tag{5.3}$$

respectively. The lattice laplacian with respect to each variable I or J is defined as

$$\nabla^2_I O^1_{IJ} = O^1_{I+1,J} - 2O^1_{IJ} + O^1_{I-1,J}\,. \tag{5.4}$$

The second term on the right hand side of eq. (5.2) is a contact term that originates when $I = J$. The problem for finding the spectrum of the operator in (5.2) is treated carefully in appendix B. There, it is found that, in the large M limit, the problem of finding the spectrum of the difference operator with $I \le J$ and the contact term can be replaced by simply looking for the spectrum of the difference operator operating on symmetric, doubly periodic functions and no contact term. For simplicity, we will implement this procedure here. To begin, we define a symmetric function

$$O_{IJ} = \begin{cases} \mathrm{TR}(A_1 \ldots \Phi_I \ldots \Phi_J \ldots A_M) & I \le J \\ \mathrm{TR}(A_1 \ldots \Phi_J \ldots \Phi_I \ldots A_M) & I > J \end{cases}$$

Then, we must take the large M limit.

Introducing the continuous variables $x = I/M$ and $y = J/M$, and taking M large, x, y take values between 0 to 1. Consequently, the continuum limit of equation (5.2) reads

$$\left(2p^- - 2\right) \circ O^1(x,y) = \frac{g_{YM}^2 N}{8\pi^2 M}\left(-\left(\partial_x^2 + \partial_y^2\right) - \delta(x-y)\left(\partial_x - \partial_y\right)\right) O^1(x,y) \tag{5.5}$$

where $\delta_{IJ} \to \delta(x-y)/M$ in the continuum limit and $2p^- = D - M$. As anticipated, if we assume that $O^1(x,y)$ is symmetric, the last term goes away in the continuum limit. Now, taking into account the fact that $O(x,y)$ is periodic in each variable with period 1, we find

$$2p^- \circ \mathcal{O}_{n_1,n_2,1} = \left[2 + \frac{1}{2}\frac{g_{YM}^2 N}{M}(n_1^2 + n_2^2)\right] O_{n_1,n_2,1}\,. \tag{5.6}$$

This result reproduces the requisite string energy spectrum (2.31) for the case of two oscillators up to one loop order. It shows that for $k = 1$ the operators corresponding to winding states with two oscillators are free string states since they get only planar corrections to the anomalous dimension. More generally this result holds independently on the number and type of impurities one considers since the single trace operator cannot be split into multi-trace operators by the action of the effective interaction hamiltonian.

6. Two impurities, $k = 2$

In this section we will study gauge theory operators with two impurity Φ fields that describe the string theory sector with discrete light-cone momentum $k = 2$. The basis of operators with two impurities is

$$\begin{aligned}
O_{IJ}^{C2} &= \mathrm{TR}(A_1 A_2 \cdots A_{I-1}\Phi_I A_I \cdots A_{J-1}\Phi_J A_J \cdots A_M A_1 \cdots A_M)\,, \qquad I \le J\\
O_{IJ}^{S2} &= \mathrm{TR}(A_1 A_2 \cdots A_{I-1}\Phi_I A_I \cdots A_M A_1 \cdots A_{J-1}\Phi_J A_J \cdots A_M)\\
O_{IJ}^{C11} &= \mathrm{TR}(A_1 A_2 \cdots A_{I-1}\Phi_I A_I \cdots A_{J-1}\Phi_J A_J \cdots A_M)\mathrm{TR}(A_1 \cdots A_M)\,, \qquad I \le J\\
O_I^1 O_J^1 &= \mathrm{TR}(A_1 A_2 \cdots A_{I-1}\Phi_I A_I \cdots A_M)\mathrm{TR}(A_1 A_2 \cdots A_{J-1}\Phi_J A_J \cdots A_M)
\end{aligned} \tag{6.1}$$

Again, there are linear combinations of these operators which are special. For example,

$$(D - 2M - 2) \circ \left(O_{IJ}^{C2} + O_{IJ}^{S2}\right) = \frac{g_{YM}^2 MN}{8\pi^2}\left(-\nabla_I^2 - \nabla_J^2 - \delta_{IJ}(\nabla_J - \hat{\nabla}_I)\right)\left(O_{IJ}^{C2} + O_{IJ}^{S2}\right). \tag{6.2}$$

It is also periodic in that

$$O_{IM+1}^{C2} + O_{IM+1}^{S2} = O_{1I}^{C2} + O_{1I}^{S2}\,.$$

We show in appendix B that, in the double scaling limit, this operator with this boundary condition has the spectrum

$$2p^- \circ \left(O_{IJ}^{C2} + O_{IJ}^{S2}\right) = \left[2 + \frac{1}{2}\frac{g_{YM}^2 N}{M}\left(n_1^2 + n_2^2\right)\right]\left(O_{IJ}^{C2} + O_{IJ}^{S2}\right). \tag{6.3}$$

In a similar way, we can see that the combination $O_{IJ}^{C11} + O_I^1 O_J^1$ is mixed with the other operators, but the mixing vanishes in the double scaling limit. It then obeys the

JHEP11(2004)053

same equation with the same boundary condition as $O^{C2}_{IJ}+O^{S2}_{IJ}$ and therefore has the same spectrum

$$2p^-\circ\left(O^{C11}_{IJ}+O^1_IO^1_J\right)=\left[2+\frac{1}{2}\frac{g_{YM}^2N}{M}\left(n_1^2+n_2^2\right)\right]\left(O^{C11}_{IJ}+O^1_IO^1_J\right). \qquad (6.4)$$

These are half of the allowed states. Recall that the two-oscillator state of the string, expanded to the leading order had spectrum

$$2p^-=2+\frac{1}{2}\frac{g_{YM}^2N}{M}\left(\left(\frac{n_1}{2}\right)^2+\left(\frac{n_2}{2}\right)^2\right)+\cdots, \qquad n_1+n_2=2\cdot\text{integer}.$$

The latter is the level matching condition. It implies that n_1 and n_2 are either both even or both odd. In the above, we have found two towers of states where they are both even. They could be associated with two string states with $k=2$ and two oscillators excited, both with even world-sheet momenta.

Finally, there are two states left. Operator mixing can be diagonalized in the double scaling limit by taking the linear combinations

$$O^+_{IJ}=O^{C2}_{IJ}-O^{S2}_{IJ}+O^{C11}_{IJ}-O^1_IO^1_J \qquad (6.5)$$

$$O^-_{IJ}=O^{C2}_{IJ}-O^{S2}_{IJ}-O^{C11}_{IJ}+O^1_IO^1_J. \qquad (6.6)$$

These have the boundary condition that

$$O^+_{IM+1}=-O^-_{1I}$$

In the double scaling limit, they obey the equations

$$(D-2M-2)\circ O^+(x,y)=\frac{g_{YM}^2MN}{8\pi^2}\left(-\partial_x^2-\partial_y^2-2\frac{M}{N}\epsilon(x-y)(\partial_x-\partial_y)\right)O^+(x,y) \qquad (6.7)$$

$$(D-2M-2)\circ O^-(x,y)=\frac{g_{YM}^2MN}{8\pi^2}\left(-\partial_x^2-\partial_y^2+2\frac{M}{N}\epsilon(x-y)(\partial_x-\partial_y)\right)O^-(x,y) \qquad (6.8)$$

We comment that these operators were originally defined only for $x\leq y$ and with a contact interaction. We have used the trick discussed in appendix B of extending the function symmetrically to all x and y so that $O^\pm(x,y)=O^\pm(y,x)$ and cancelling the contact interaction. (We can check after we have found a solution that the contact term operating on it indeed vanishes.) The boundary conditions are then

$$O^+(x+1,y)=O^+(x,y+1)=O^-(x,y)\quad,\quad O^-(x+1,y)=O^-(x,y+1)=O^+(x,y).$$

These are compatible with the above equation if we extend the antisymmetric step function so that

$$\begin{aligned}\epsilon(x)&=1\,,\quad x\in(-2,1)\,,\quad(0,1)\,,\quad(2,3),\ldots\\ \epsilon(x)&=-1\,,\quad x\in(-1,0)\,,\quad(1,2)\,,\quad(3,4),\ldots\end{aligned} \qquad (6.9)$$

Then $\epsilon(x\pm1)=-\epsilon(x)$.

It is natural to introduce the center of mass and the relative coordinates R and r defined as

$$R = \frac{x+y}{2}\,, \qquad r = y - x \tag{6.10}$$

where $0 \leq R \leq 1$ and $-1 \leq r \leq 1$. Then, the variables separate and we can make the ansatz

$$O^{+}(R,r) = e^{2\pi i m R} u(r)\,, \qquad O^{-}(R,r) = e^{2\pi i m R} v(r)$$

where, since the functions should be periodic in R, m is an integer.

The eigenvalues of the dilatation operator are

$$D = 2M + 2 + \frac{g_{YM}^2 N}{8\pi^2 M}\left(\lambda + 2\pi^2 m^2\right)$$

where λ are eigenvalues obtained by solving the equations

$$\begin{aligned} \left(\partial_r^2 - 2g_2\epsilon(r)\partial_r + \frac{\lambda}{2}\right) u(r) &= 0 \\ \left(\partial_r^2 + 2g_2\epsilon(r)\partial_r + \frac{\lambda}{2}\right) v(r) &= 0\,. \end{aligned} \tag{6.11}$$

The equation should now be solved with the boundary condition

$$u(r+1) = -(-1)^m v(r)\,. \tag{6.12}$$

This boundary condition has already been used to set the eigenvalues equal in (6.11). We have also denoted

$$g_2 = \frac{M}{N}\,. \tag{6.13}$$

Consider the equation for the function $u(r)$ (the equation for $v(r)$ is identical with $g_2 \to -g_2$)

$$\begin{cases} \left(\partial_r^2 - 2g_2\partial_r + \frac{\lambda}{2}\right) u(r) = 0\,, & r > 0 \\ \left(\partial_r^2 + 2g_2\partial_r + \frac{\lambda}{2}\right) u(r) = 0\,, & r < 0\,. \end{cases} \tag{6.14}$$

The solution of this one dimensional eigenvalue problem is trivial. The solution for positive r can be written as

$$u(r)_+ = a e^{\omega_+ r} + b e^{\omega_- r} \tag{6.15}$$

where $\omega_\pm = g_2 \pm \sqrt{g_2^2 - \frac{\lambda}{2}}$. For negative r one has

$$u(r)_- = c e^{\omega'_+ r} + d e^{\omega'_- r} \tag{6.16}$$

where $\omega'_\pm = -g_2 \pm \sqrt{g_2^2 - \lambda/2}$.

By requiring the continuity of the function and its first derivative in $r=0$ and the continuity of the function in $r=1$ we find

$$u(r)=a\begin{cases}\left(\sqrt{g_2^2-\frac{\lambda}{2}}-g_2\right)e^{(g_2+\sqrt{g_2^2-\frac{\lambda}{2}})r}+\left(\sqrt{g_2^2-\frac{\lambda}{2}}+g_2\right)e^{(g_2-\sqrt{g_2^2-\frac{\lambda}{2}})r} & r>0\\ \left(\sqrt{g_2^2-\frac{\lambda}{2}}+g_2\right)e^{(-g_2+\sqrt{g_2^2-\frac{\lambda}{2}})r}+\left(\sqrt{g_2^2-\frac{\lambda}{2}}-g_2\right)e^{-(g_2+\sqrt{g_2^2-\frac{\lambda}{2}})r} & r<0\,.\end{cases} \tag{6.17}$$

where a is a normalization constant that we will keep undetermined. Then we should impose the boundary conditions on the function and its derivative. The function $u(r)$ and its derivative must be periodic of period 2. Since $u(r)$ is symmetric $u(1)=u(-1)$, but requiring that $u'(1)=u'(-1)$ implies that $u'(1)=0$. The latter condition leads to

$$\frac{\lambda}{2}\left(e^{-\sqrt{g_2^2-\frac{\lambda}{2}}}-e^{\sqrt{g_2^2-\frac{\lambda}{2}}}\right)=0 \tag{6.18}$$

which determines the eigenvalue

$$\lambda=\begin{cases}0\\ 2g_2^2+2\pi^2n^2\end{cases} \tag{6.19}$$

the solution $\lambda=0$ has to be discarded because to it corresponds a constant function $u(r)$ which is not compatible with the boundary condition (6.12). Moreover n has to be different from zero. In fact, if we take n to be zero, then $\lambda=2g_2^2$. But this solution for λ implies $u(r)=0$ as can be easily seen from equation (6.17). Since the eigenvalue depends only on g_2^2 the same eigenvalue will be found for the solution $v(r)$. Finally, the light-cone momenta for these two states are

$$2p^-=2+\frac{g_{YM}^2N}{8\pi^2M}\left(2\pi^2n^2+2\pi^2m^2+2g_2^2\right).$$

Now, if we take $m=\frac{1}{2}(n_1+n_2)$ and $n=\frac{1}{2}(n_1-n_2)$ where $n_1\pm n_2$ are necessarily even, we get

$$2p^-=2+\frac{1}{2}\frac{g_{YM}^2N}{M}\left(n_1^2+n_2^2\right)+\frac{g_{YN}^2M}{4\pi^2N}\,.$$

We must still impose the boundary condition (6.12). To get $v(r)$, we change the sign of g_2 in $u(r)$. Then, we see that there are two possibilities for the integers m and n: either m is even and n is odd, or m is odd and n is even. In either case, this implies that both n_1 and n_2 are odd integers.

Let us review: Level matching, $n_1+n_2=2\cdot$integer, requires that the world-sheet momenta n_1 and n_2 are either both even integers or both odd integers. When they are both even, the free string spectra are not corrected by string loops, at least to the leading order in world-sheet momenta. When they are both odd, they are corrected by string loops. One way to present the correction is to rename the constant which governs the world-sheet energy

$$\tilde{\alpha}=\frac{g_{YM}^2N}{M}\,.$$

Then we can write the above formula as

$$2p^- = 2 + \frac{1}{2}\tilde{\alpha}\left(n_1^2 + n_2^2\right) + \frac{4}{\tilde{\alpha}} g_s^2 \,. \tag{6.20}$$

This is an exact result for string states with two units of light-cone momentum, $k = 2$. As expected, the first two terms on the right-hand-side of (6.20) give the free string spectrum. What is surprising is that the interaction term truncates at second order in the closed string coupling. In principle, the eigenvalue could have been a complicated function of $g_2 = N/M = 4\pi g_s/\tilde{\alpha}$ and could have generated all orders in the string coupling. We do not presently have an understanding of why it should truncate at second order. This truncation is a definite prediction for string loop corrections which should be checked in string theory.

7. Two impurities, $k = 3$

In this section we will consider the gauge theory operators with two insertion of Φ fields that describe the string theory sector with DLCQ momentum $k = 3$. The basis of operators with two impurities is made of the following 9 operators

$$O^{S3}_{IJ}\,,\quad O^{S3}_{JI}\,,\quad O^{C3}_{IJ}\,,\qquad O^{C21}_{IJ}\,,\quad O^{S21}_{IJ}\,,\quad O^{C12}_{IJ}\,,\quad O^{C111}_{IJ}\,,\quad O^2_I O^1_J\,,\quad O^1_I O^{11}_J \tag{7.1}$$

where

$$\begin{aligned}
O^{C3}_{IJ} &= \mathrm{TR}[A_1A_2\cdots A_{I-1}\Phi_I A_I\cdots A_{J-1}\Phi_J A_J\cdots A_M(A_1\cdots A_M)^2]\,,\quad I\le J\\
O^{S3}_{IJ} &= \mathrm{TR}[A_1A_2\cdots A_{I-1}\Phi_I A_I\cdots A_M A_1\cdots A_{J-1}\Phi_J A_J\cdots A_M A_1\cdots A_M]\\
O^{C21}_{IJ} &= \mathrm{TR}[A_1A_2\cdots A_{I-1}\Phi_I A_I\cdots A_{J-1}\Phi_J A_J\cdots A_M A_1\cdots A_M]\mathrm{TR}[A_1\cdots A_M]\,,\quad I\le J\\
O^{S21}_{IJ} &= \mathrm{TR}[A_1A_2\cdots A_{I-1}\Phi_I A_I\cdots A_M A_1\cdots A_{J-1}\Phi_J A_J\cdots A_M]\mathrm{TR}[A_1\cdots A_M]\,,\quad I\le J\\
O^{C12}_{IJ} &= \mathrm{TR}[A_1A_2\cdots A_{I-1}\Phi_I A_I\cdots A_{J-1}\Phi_J A_J\cdots A_M]\mathrm{TR}[(A_1\cdots A_M)^2]\,,\quad I\le J\\
O^{C111}_{IJ} &= \mathrm{TR}(A_1A_2\cdots A_{I-1}\Phi_I A_I\cdots A_{J-1}\Phi_J A_J\cdots A_M)\left(\mathrm{TR}[A_1\cdots A_M]\right)^2\,,\quad I\le J\\
O^2_I O^1_J &= \mathrm{TR}[A_1A_2\cdots A_{I-1}\Phi_I A_I\cdots A_M A_1\cdots A_M]\mathrm{TR}[A_1A_2\cdots A_{J-1}\Phi_J A_J\cdots A_M]\\
O^1_I O^{11}_J &= \mathrm{TR}[A_1A_2\cdots A_{I-1}\Phi_I A_I\cdots A_M]\mathrm{TR}[A_1A_2\cdots A_{J-1}\Phi_J A_J\cdots A_M]\mathrm{TR}[A_1\cdots A_M]\,.
\end{aligned} \tag{7.2}$$

As explained in appendix B, in the large M limit, the extension to all I and J of the functions O_{IJ} defined only for $I \le J$, becomes a symmetric doubly periodic function. There are linear combinations of these states for which the action of $(D - 3M - 2)$ reduces to that of the laplacian. Two states of this type which are periodic of period 1 are

$$\begin{aligned}
u^1_{IJ} &= O^{C21}_{IJ} + O^{S21}_{IJ} + \frac{1}{2}\left(O^1_I O^2_J + O^2_I O^1_J\right)\\
u^2_{IJ} &= \frac{1}{2} O^{C111}_{IJ} + O^1_I O^{11}_J \,.
\end{aligned} \tag{7.3}$$

These state are periodic in that

$$u^1_{IM+1} = u^1_{1I}\,,\qquad u^2_{IM+1} = u^2_{1I}\,. \tag{7.4}$$

In the double scaling limit these operators that satisfy the boundary condition (7.4), have the spectrum

$$2p^- \circ u^i(x,y) = \left(2 - \frac{g^2_{YM}N}{8\pi^2 M}\nabla^2\right)u^i(x,y) = \left(2 + \frac{g^2_{YM}N}{8\pi^2 M}\lambda\right)u^i_{IJ}\,.$$

To get a string state corresponding to 3 units of light-cone momentum, the solution to this equation has to be put into the form $\exp[2i\pi(n_1 x + n_2 y)/3]$ as in (2.38). As in the $k = 2$ case however, since these states must be periodic of period 1 both in x and y, n_1 and n_2 must be multiples of 3. λ then is given by $\lambda = 4\pi^2(n_1^2 + n_2^2)/9$, providing an anomalous dimension for these operators $\Delta = g_{YM}^2 N(n_1^2 + n_2^2)/(18M)$. This is again an exact result, the string states corresponding to these operators are free, the one-loop anomalous dimension provides in fact the expansion to order g_{YM}^2 of the free string spectrum. Δ does not get corrections beyond the planar level namely from string interactions. However, only string states created by oscillators with n_1 and n_2 multiples of 3 behave as free string states.

We are left with 7 independent states, which can be reorganized in a more convenient way by introducing the following combinations of double and triple trace operators

$$\begin{aligned} D_{IJ}^{12} &= O_{IJ}^{C12} - O_I^1 O_J^2 - O_I^2 O_J^1 \\ T_{IJ}^{111} &= O_{IJ}^{C111} - O_I^1 O_J^{11}\,. \end{aligned} \tag{7.5}$$

These combinations are orthogonal to (7.3) and together with the operators O_{IJ}^{S3}, O_{JI}^{S3}, O_{IJ}^{C3}, O_{IJ}^{C21}, O_{IJ}^{S21} form a closed basis for $(D - 3M - 2)$. Consider a general continuous symmetric linear combination of these states and call it u_{IJ}

$$\begin{aligned} u_{IJ} = \theta(J - I)\left[c_1 O_{IJ}^{S3} + c_2 O_{JI}^{S3} + c_3 O_{IJ}^{C3} + c_4 O_{IJ}^{C21} + c_5 O_{IJ}^{S21} + c_6 D_{IJ}^{12} + c_7 T_{IJ}^{111}\right] + \\ +\theta(I - J)\left[c_2 O_{IJ}^{S3} + c_1 O_{JI}^{S3} + c_3 O_{IJ}^{C3} + c_4 O_{IJ}^{C21} + c_5 O_{IJ}^{S21} + c_6 D_{IJ}^{12} + c_7 T_{IJ}^{111}\right] \end{aligned} \tag{7.6}$$

where we have introduced the Heaviside theta function defined as

$$\theta(J - I) = \begin{cases} 1 & J > I \\ 1/2 & J = I \\ 0 & J < I\,. \end{cases}$$

Acting with $(D - 3M - 2)$ on u_{IJ} and taking the double scaling limit, one discovers that the system of equations closes after three iteration of the action of $(D - 3M - 2)$, namely

$$\begin{cases} (D - 3M - 2) \circ u(x, Y) = \dfrac{g_{YM}^2 N}{8\pi^2 M}\left(-\nabla^2 u(x, y) - \dfrac{M}{N}\epsilon(x - y)\left(\partial_x - \partial_y\right) w(x, y)\right) \\ (D - 3M - 2) \circ w(x, y) = \dfrac{g_{YM}^2 N}{8\pi^2 M}\left(-\nabla^2 w(x, y) - \dfrac{M}{N}\epsilon(x - y)\left(\partial_x - \partial_y\right) v(x, y)\right) \\ (D - 3M - 2) \circ v(x, y) = \dfrac{g_{YM}^2 N}{8\pi^2 M}\left(-\nabla^2 v(x, y) - 9\dfrac{M}{N}\epsilon(x - y)\left(\partial_x - \partial_y\right) w(x, y)\right) \end{cases} \tag{7.7}$$

where $w(x, y)$ and $v(x, y)$ are written in terms of the coefficients of $u(x, y)$ in appendix C and the step function $\epsilon(x - y)$ has to be periodically continued as in (6.9). It is not difficult now to decouple this system of equations. There is in fact a solution to which correspond free string states. It is given by the solutions of the equations

$$w(x, y) = v(x, y) = 0\,, \qquad (D - 3M - 2) \circ u(x, y) = -\frac{g_{YM}^2 N}{8\pi^2 M}\nabla^2 u(x, y)\,. \tag{7.8}$$

The first two equations in (7.8) fix the values of the coefficients c_i to give for $u(x,y)$ three independent solutions corresponding to three states periodic of period 1 both in x and y

$$\begin{aligned} u^3(x,y) &= O^{S3}(x,y) + O^{S3}(y,x) + O^{C3}(x,y) \\ u^4(x,y) &= \theta(y-x)\left[O^{S3}(y,x) + O^{C3}(x,y) - \frac{1}{3}T^{111}(x,y)\right] + \\ &\quad +\theta(x-y)\left[O^{S3}(x,y) + O^{C3}(x,y) - \frac{1}{3}T^{111}(x,y)\right] \\ u^5(x,y) &= O^{C21}(x,y) + O^{S21}(x,y) - \frac{1}{4}D^{12}(x,y)\,. \end{aligned} \tag{7.9}$$

These states have the same free string spectrum of u^1 and u^2. Since they also have the same periodicity one has to require the same condition on the oscillators generating these string states, namely that the levels to which they correspond (n_1 and n_2) must be multiples of 3.

We have still to find four states. Taking the linear combinations

$$\psi_\pm(x,y) = 3w(x,y) \pm v(x,y) \tag{7.10}$$

from (7.7), one gets the equations

$$\begin{cases} (D-3M-2)\circ\psi_+(x,y) = \dfrac{g_{YM}^2 N}{8\pi^2 M}\left(-\nabla^2\psi_+(x,y) - 3\dfrac{M}{N}\epsilon(x-y)\left(\partial_x-\partial_y\right)\psi_+(x,y)\right) \\ (D-3M-2)\circ\psi_-(x,y) = \dfrac{g_{YM}^2 N}{8\pi^2 M}\left(-\nabla^2\psi_-(x,y) + 3\dfrac{M}{N}\epsilon(x-y)\left(\partial_x-\partial_y\right)\psi_-(x,y)\right) \\ (D-3M-2)u(x,y) = \dfrac{g_{YM}^2 N}{8\pi^2 M}\left(-\nabla^2 u(x,y) - \dfrac{M}{6N}\epsilon(x-y)\left(\partial_x-\partial_y\right)(\psi_+ + \psi_-)\right) \end{cases} \tag{7.11}$$

where the first two equations are decoupled. In appendix C we show that in both $\psi_\pm$ there are only two independent coefficients of the original 9 basis operators (7.1). Therefore $\psi_\pm$ provide the remaining 4 states. As we will see in the next section these states correspond to operators that are periodic of period 6 and get computable corrections to all orders in the genus expansion.

7.1 Solution

In this subsection we solve the equations (7.11) for $\psi_\pm(x,y)$. To this purpose, it is convenient to introduce again the center of mass and relative coordinates (6.10). Let us focus on $\psi_+(R,r)$. The variables separate and we can write

$$\psi_+(R,r) = e^{2i\pi m R}\psi_+(r)$$

where m is an integer. The eigenvalues of the dilatation operator are

$$D = 3M + 2 + \frac{g_{YM}^2 N}{8\pi^2 M}\left(\lambda + 2\pi^2 m^2\right). \tag{7.12}$$

The eigenvalues λ can be obtained by solving the equation

$$\left(\partial_r^2 - 3g_2\epsilon(r)\partial_r + \frac{\lambda}{2}\right)\psi_+(r) = 0 \tag{7.13}$$

where g_2 is defined in (6.13). The states we are considering are in general periodic of period 6 namely when $r \to r+6$ (corresponding to $J \to J+6M$) ψ_+ goes to itself $\psi_+(r+6) = \psi_+(r)$. The relative coordinate has a range $-1 \le r \le 1$ but it can be periodically continued to the range $-3 \le r \le 3$ in order to realize the correct periodicity for the functions $\psi_\pm(r)$.

We divide the interval between $r = -3$ and $r = 3$ in six regions and we impose that the solution of the equation (7.13) matches at the boundary of each region. We also can consider only half of the total region, namely the region $0 \le r \le 3$, because the function $\psi_+(r)$ is symmetric for $r \to -r$.

In the regions I, $0 \le r \le 1$, the solution of the equation (7.13) can be written as

$$\psi_+(r) = a\left(\omega_- e^{\omega_+ r} - \omega_+ e^{\omega_- r}\right) \tag{7.14}$$

where $\omega_\pm = \frac{3}{2}g_2 \pm \sqrt{\frac{9}{4}g_2^2 - \frac{\lambda}{2}}$. In the region II, $1 \le r \le 2$, we write the solution as

$$\psi_+(r) = be^{\eta_+ r} + ce^{\eta_- r} \tag{7.15}$$

where $\eta_\pm = -\frac{3}{2}g_2 \pm \sqrt{\frac{9}{4}g_2^2 - \frac{\lambda}{2}}$. For $r = 1$ we have to require the continuity of the function

$$a\left(\omega_- e^{\omega_+} - \omega_+ e^{\omega_-}\right) = be^{\eta_+} + ce^{\eta_-} \tag{7.16}$$

so that, in the region II, we get

$$\psi_+(r) = \frac{ae^{\eta_+(r-2)}}{\sqrt{9g_2^2 - 2\lambda}}\left[\lambda e^{\sqrt{9g_2^2-2\lambda}} - 3g_2\omega_+\right] + \frac{ae^{\eta_-(r-2)}}{\sqrt{9g_2^2 - 2\lambda}}\left[\lambda e^{-\sqrt{9g_2^2-2\lambda}} - 3g_2\omega_-\right]. \tag{7.17}$$

Finally, in the region III, $2 \le r \le 3$, we write

$$\psi_+(r) = de^{\omega_+ r} + fe^{\omega_- r}\,. \tag{7.18}$$

Requiring again the continuity of the function and of its derivative in $r = 2$, we have that in the region III,

$$\begin{aligned}\psi_+(r) \;=\; & \frac{ae^{\omega_+(r-2)}}{(9g_2^2 - 2\lambda)}\left[9g_2^2\omega_- + 3g_2\lambda\left(1 - e^{-\sqrt{9g_2^2-2\lambda}}\right) - 2\lambda\omega_- e^{\sqrt{9g_2^2-2\lambda}}\right] - \\ & - \frac{ae^{\omega_-(r-2)}}{(9g_2^2 - 2\lambda)}\left[9g_2^2\omega_+ + 3\lambda g_2\left(1 - e^{\sqrt{9g_2^2-2\lambda}}\right) - 2\lambda\omega_+ e^{-\sqrt{9g_2^2-2\lambda}}\right].\end{aligned} \tag{7.19}$$

The function $\psi_+(r)$, by construction, is periodic of period 6 and is symmetric for $r \to -r$. By requiring that $\psi_+'(3) = \psi_+'(0) = 0$, we get an exact transcendental equation for λ

$$\frac{\lambda e^{-3(-g_2+\sqrt{9g_2^2-2\lambda})/2}}{18g_2^2 - 4\lambda}\left(1 - e^{\sqrt{9g_2^2-2\lambda}}\right)\left(2\lambda(1 + e^{2\sqrt{9g_2^2-2\lambda}}) + e^{\sqrt{9g_2^2-2\lambda}}(2\lambda - 27g_2^2)\right) = 0 \tag{7.20}$$

The differential equation for ψ_- can be obtained from the one for ψ_+ by changing the sign of g_2. Moreover, the eigenvalue equation, as it should, depends only on g_2^2. Consequently, one gets an identical transcendental relation for the eigenvalue λ_- by solving the equation for ψ_-.

JHEP11(2004)053

The values of λ that solve this equation are $\lambda = 0$, $\lambda = 2\pi^2 n^2 + \frac{9}{2} g_2^2$ or the solutions of

$$f(g_2^2, \lambda) = 4\lambda \cosh \sqrt{9g_2^2 - 2\lambda} + 2\lambda - 27g_2^2 = 0 \tag{7.21}$$

$\lambda = 0$ has to be discarded because to it corresponds a trivial constant solution. $\lambda = 2\pi^2 n^2 + \frac{9}{2} g_2^2$ corresponds to states with period 2 in the variable r. These states cannot be realized with linear combinations of the form $\psi_\pm$ which are necessarily periodic of period 6. Thus if we require the periodicity allowed for the operators of the form $\psi_\pm$ one has to consider only the solutions of (7.21).

From (7.21) one could in principle compute the energy eigenvalues of the gauge theory to all orders in g_2^2 and verify that they are in agreement with those obtained from the dual string energy spectrum.

To obtain a genus expansion solution of the string spectrum one can write the eigenvalue λ as a Taylor series in powers of g_2^2 around $g_2 = 0$

$$\lambda = \lambda_0 + \left.\frac{d\lambda}{dg_2^2}\right|_{\lambda_0, g_2=0} g_2^2 + \frac{1}{2} \left.\frac{d^2\lambda}{(dg_2^2)^2}\right|_{\lambda_0, g_2=0} g_2^4 + \frac{1}{3!} \left.\frac{d^3\lambda}{(dg_2^2)^3}\right|_{\lambda_0, g_2=0} g_2^6 + \cdots \tag{7.22}$$

where λ_0 is the eigenvalue for $g_2 = 0$. From (7.21) with $g_2 = 0$ one finds that $\lambda_0 = 2\pi^2 n^2/9$ with n any integer different from zero which is not multiple of 3.

The derivatives of λ as a function of g_2^2 can be obtained by taking the derivatives of the implicit function (7.21) and computing them at $\lambda = \lambda_0$ and $g_2 = 0$. In fact since $f(g_2^2, \lambda) = 0$ also its differential must vanish

$$\frac{\partial f}{\partial g_2^2} dg_2^2 + \frac{\partial f}{\partial \lambda} d\lambda = 0 \,. \tag{7.23}$$

From this equation one finds

$$\frac{d\lambda}{dg_2^2} = -\frac{\partial f}{\partial g_2^2} \left(\frac{\partial f}{\partial \lambda}\right)^{-1} \tag{7.24}$$

and from the last result we can compute all the coefficients of the equation (7.22).

To the sixth order in g_2 the eigenvalue is

$$\begin{aligned}
\lambda = {} & \frac{2}{9}\pi^2 n^2 + \frac{9\left(2\pi n \sin\frac{2\pi n}{3} - 9\right)}{2\left[2\pi n \sin\frac{2\pi n}{3} - 9 + 12\left(\sin\frac{\pi n}{3}\right)^2\right]} g_2^2 + \\
& + \frac{2187 (\sin\frac{\pi n}{3})^3 \left(6\pi n \sin\frac{\pi n}{3} - 33\cos\frac{\pi n}{3} - 3(-1)^n\right)}{2\pi n \left(2\pi n \sin\frac{2\pi n}{3} - 3 - 6\cos\frac{2\pi n}{3}\right)^3} g_2^4 + \\
& + \Bigg\{ -2484\pi n - 80\pi^3 n^3 + 405\left(\sin\frac{2\pi n}{3} + \sin\frac{4\pi n}{3}\right) + \\
& \quad + 2\pi n \Bigg[(-243 + 4\pi^2 n^2) 5 \cos\frac{2\pi n}{3} + 27 + \\
& \qquad + \pi n \left(4\pi n + 16\pi n \cos\frac{4\pi n}{3} + 3\left(173 \sin\frac{2\pi n}{3} + 38 \sin\frac{4\pi n}{3}\right)\right)\Bigg]\Bigg\} \times \\
& \times \frac{59049 \left(\sin\frac{\pi n}{3}\right)^4}{8\pi^3 n^3 \left(2\pi n \sin\frac{2\pi n}{3} - 3 - 6\cos\frac{2\pi n}{3}\right)^5} g_2^6 + \cdots
\end{aligned} \tag{7.25}$$

where n is related to the world-sheet momenta of the string excitations by $n = n_1 - n_2$. We remind here that n has to be different from zero and not a multiple of 3 otherwise λ_0 would not be a solution of the transcendental equation (7.21) for $g_2 = 0$. If it was a multiple of 3, $n = 3l$, first of all the solution (7.25) would collapse into the one previously analyzed $\lambda = 2\pi^2 l + \frac{9}{2} g_2^2$ which does not have the correct periodicity. Moreover, in this case, since $n_1 + n_2 = 3m$ by the level matching condition, both n_1 and n_2 would be multiples of 3. For these string states however, we have already shown that the predictions of the gauge theory are that there are no string loop corrections to the free string spectrum. Summarizing, when the world-sheet momenta n_1 and n_2 of the string excitations are both multiples of 3, the gauge theory predicts a free string spectrum. When they are not multiples of 3, the anomalous dimension of the corresponding gauge theory operators gets corrections to all non planar orders. For example the spectrum of the two-oscillator state of the string, expanded up to genus three is given by

$$2p^- = 2 + \frac{g_{YM}^2 N}{8\pi^2 M} \left(2\pi^2 m^2 + \lambda + + O(g_2^8) \right). \tag{7.26}$$

To match the string spectrum in this case, here $m = (n_1 + n_2)/3$ (this establishes the level matching condition) and λ up to the sixth order in g_2 is given in (7.25) with $n = n_1 - n_2$. The procedure can be iterated to obtain also higher order corrections to this spectrum.

8. Summary and conclusions

In this paper we have studied non planar corrections to the spectrum of operators in the $\mathcal{N} = 2$ supersymmetric Yang Mills theory which are dual to string states in the maximally supersymmetric pp-wave background with a compact light-cone direction. The existence of a discrete light-cone momentum simplifies the calculations of the anomalous dimension of gauge theory operators dual to string states.

The gauge theory predictions in the double scaling limit of large-N and long operators $M \to \infty$, with M/N fixed, are:

- String states with one oscillator and any value of the light cone momentum k are quasi-protected in that they have an anomalous dimension which does not have corrections beyond the planar level, their string spectrum is free. This is a correct prediction until k is of order N (or M), which is when operator mixing sets in.

- String states with one unit of light cone momentum and any number of oscillators are free states in that they do not get string loop corrections.

- For string states with two units of light cone momentum and two impurities there are two possibilities: states for which both the world-sheet momenta are integer multiples of $k = 2$, namely are even, have a free spectrum; states for which both the world-sheet momenta are odd get only the one string loop correction given in (6.20). The states with even-odd world-sheet momenta are excluded by level matching.

JHEP11(2004)053

- String states with three units of light cone momentum and two impurities for which both the world-sheet momenta are integer multiples of $k = 3$, have a free spectrum. $k = 3$ states for which both the world-sheet momenta are not integer multiples of 3 get computable corrections to all orders, as in (7.26), where λ is a solution of the exact equation (7.21). Up to three loop in the effective genus counting parameter expansion, λ is given in (7.25).

The AdS/CFT correspondence should be checked wherever possible. Recently it were found some discrepancies in the energy spectrum of BMN states with two string excitations, computed in the framework of light-cone string field theory [32], and that computed from the gauge theory [20, 40, 41]. Even if such discrepancies might be originated by the choice of the cubic vertex of IIB strings in a pp-wave background [42, 43], it would be extremely important to check these results also in other contexts. The DLCQ of the string greatly simplifies the setting where the duality is realized, therefore it might help in solving, one way or the other, for such discrepancies. The calculations presented in this paper are precisely for the case of two string excitations were the discrepancies were found in the usual pp-wave correspondence.

Acknowledgments

G.G. thanks A. Lerda for an interesting discussion, G.G. and M.O. are grateful to S. Mukhi for useful suggestions.

A. Anomalous dimensions

In this appendix, we will compute the divergent parts which appear in the one-loop corrections to the two-point function of composite operators made from products of scalar fields. These loop corrections arise from the Wick contractions of the operators in the interaction lagrangian with the operators inside the composites. Here, we will assume that the interaction lagrangian is quartic and has no derivatives. We will also assume that there are no derivatives in the composite operators. Finally, we assume that there are only two contractions between the interaction lagrangian and each composite operator. Then, the one-loop correction to the two-point function is

$$-\left\langle \mathcal{O}_\alpha(y)\mathcal{L}_F(x)\bar{\mathcal{O}}_{\bar\beta}(0)\right\rangle = -\left\langle \mathcal{O}_\alpha H\bar{\mathcal{O}}_{\bar\beta}\right\rangle_{MM}[\Delta(y)]^{\frac{\Delta^0_\alpha+\Delta^0_\beta}{2}}\int d^4x\frac{\Delta^2(y-x)\Delta^2(x)}{\Delta^2(y)} \tag{A.1}$$

where $\Delta(y)$ is the scalar propagator. The product $\Delta^2(y-x)$ comes from the two contractions between the interaction and $\mathcal{O}_\alpha$ and $\Delta^2(x)$ from contractions with $\bar{\mathcal{O}}_{\bar\beta}$. The combinatorics of how the contraction is done are summarized in the reduced matrix element $\left\langle \mathcal{O}_\alpha H\bar{\mathcal{O}}_{\bar\beta}\right\rangle_{MM}$. We shall use dimensional regularization and work in 2ω-dimensions where the scalar propagator is

$$\Delta(x) = \frac{\Gamma(\omega-1)}{4\pi^\omega}\frac{1}{[x^2]^{\omega-1}}$$

The divergent one-loop correction has the form

$$
\begin{aligned}
&\mu^{4-2\omega}\int d^{2\omega}x\frac{\Delta^2(y-x)\Delta^2(x)}{\Delta^2(y)}\times\\
&\times\mu^{4-2\omega}\left(\frac{\Gamma(\omega-1)}{4\pi^\omega}\right)^2\frac{\Gamma(4\omega-4)}{\Gamma^2(2\omega-2)}\int_0^1 d\alpha\alpha^{2\omega-3}(1-\alpha)^{2\omega-3}\int d^{2\omega}x\frac{[y^2]^{2\omega-2}}{[x^2+y^2\alpha(1-\alpha)]^{4\omega-4}}=\\
&=\mu^{4-2\omega}\left(\frac{\Gamma(\omega-1)}{4\pi^\omega}\right)^2\frac{\Gamma(4\omega-4)}{\Gamma^2(2\omega-2)}\int_0^1 d\alpha\alpha^{2\omega-3}(1-\alpha)^{2\omega-3}\pi^\omega\frac{\Gamma(3\omega-4)}{\Gamma(4\omega-4)}\frac{[y^2]^{2\omega-2}}{[y^2\alpha(1-\alpha)]^{3\omega-4}}\\
&=\frac{1}{[\mu^2y^2]^{\omega-2}}\frac{\Gamma^2(\omega-1)\Gamma(3\omega-4)}{16\pi^\omega\Gamma^2(2\omega-2)}\frac{\Gamma^2(2-\omega)}{\Gamma(4-2\omega)}\\
&=\left(\frac{1}{8\pi^2(2-\omega)}+\frac{\ln(\mu^2y^2)}{8\pi^2}+\cdots\right)
\end{aligned}
\tag{A.2}
$$

where μ is the renormalization scale. The first, divergent term in the final expression must be subtracted using a counterterm. The remaining logarithmic term shifts the exponent in the space-dependence of the two-point function and thus contributes to the anomalous dimension. The terms denoted by dots are finite and y-independent.

To do the above integrals, we have used the Feynman parameter formula

$$
\frac{1}{A_1^{n_1}A_2^{n_2}}=\frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)}\int_0^1 d\alpha\alpha^{n_1-1}(1-\alpha)^{n_2-1}
$$

and the integral formulae

$$
\begin{aligned}
\int d^{2\omega}x[x^2+m^2]^{-s}&=\pi^\omega\frac{\Gamma(s-\omega)}{\Gamma(s)}[m^2]^{\omega-s}\\
\int_0^1 d\alpha\alpha^{\mu-1}(1-\alpha)^{\nu-1}&=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}\,.
\end{aligned}
$$

The reduced matrix element is found by taking the most singular terms in the operator product expansion

$$
\left\langle\mathcal{O}_\alpha H\bar{\mathcal{O}}_\beta\right\rangle=D_\alpha^{\ \gamma}\left\langle\mathcal{O}_\gamma\bar{\mathcal{O}}_\beta\right\rangle
$$

where

$$
:\mathcal{O}_\alpha(y)::\mathcal{L}_F(x):=\Delta^2(x-y)D_\alpha^{\ \beta}\mathcal{O}_\beta(y)+\text{less singular}\,. \tag{A.3}
$$

To one-loop order, $D_\alpha^{\ \beta}$ is the dilatation operator. Its eigenvalues are the one-loop contribution to the conformal dimensions of operators and its eigen-vectors are the operators.

B. Contact terms in the planar limit

In this appendix we will consider the effect of contact terms which appear in the leading order, planar interactions. We will show that the contact terms do not vanish in the continuum limit. However, they can be taken into account the particular symmetric prescription for two-impurity operators which is used in this paper.

Here, we consider the composite operator

$$
\mathcal{O}_{IJ}(x)=\mathrm{TR}\left(A_1(x)\ldots A_{I-1}(x)\Phi_I(x)A_I(x)\ldots A_{J-1}(x)\Phi_J(x)A_J(x)\ldots A_M(x)\right).
$$

Hereafter, we shall drop the coordinate dependence, as it should always be clear from the context that the composite operators in question are defined by products of fields at the same point. The hamiltonian acts on this operator as

$$
\begin{aligned}
H_0\mathcal{O}_{IJ} &= \frac{g_{YM}^2 MN}{8\pi^2}\left[\mathcal{O}_{I+1J}+\mathcal{O}_{I-1J}-2\mathcal{O}_{IJ}+\mathcal{O}_{IJ+1}+\mathcal{O}_{IJ-1}-2\mathcal{O}_{IJ}\right] \quad 1\le I<J\le M \\
H_0\mathcal{O}_{II} &= \frac{g_{YM}^2 MN}{8\pi^2}\left[\mathcal{O}_{I-1I}+\mathcal{O}_{II+1}-2\mathcal{O}_{II}\right], \quad I=J\,.
\end{aligned} \tag{B.1}
$$

It is augmented by the boundary condition

$$
\mathcal{O}_{IM+1}=\mathcal{O}_{1I}\,. \tag{B.2}
$$

This resembles the problem of two free particles moving on a latticized circle with a contact interaction. The contact term is important and does not vanish in the continuum limit, so must be treated with some care.

We seek an eigenstate of the operator defined by (B.1), (B.1) and (B.2). The operator in (B.1) is just the lattice laplacian which has eigenstate

$$
\psi_{IJ}(p,\ell)=e^{ipI+i\ell J}+S(p,\ell)e^{i\ell I+ipJ} \tag{B.3}
$$

and eigenvalue

$$
E=e^{ip}+e^{-ip}-2+e^{i\ell}+e^{-i\ell}-2\,. \tag{B.4}
$$

Here, we have noted that the eigenvalue problem is formally symmetric under interchanging I and J, so the eigenvalues should carry a (perhaps projective) representation of the permutation group. This representation is characterized by the 2-body S-matrix, $S(p,\ell)$, familiar from the Bethe Ansatz for spin chains. As we shall see shortly, $S(p,\ell)$ is determined by the boundary condition (B.2) and the contact term (B.1) in the hamiltonian. Requiring that the wave-function is an eigenstate of the contact interaction with the same eigenvalue leads to the Bethe equation:

$$
\left(e^{i\frac{(p+\ell)}{2}}+e^{-i\frac{(p+\ell)}{2}}-2e^{i\frac{(p-\ell)}{2}}\right)+\left(e^{i\frac{(p+\ell)}{2}}+e^{-i\frac{(p+\ell)}{2}}-2e^{-i\frac{(p-\ell)}{2}}\right)S(p,\ell)e^{i(p-\ell)}=0\,. \tag{B.5}
$$

To proceed, we need to impose the boundary condition (B.2). It implies

$$
e^{ipI+i\ell(M+1)}+S(p,\ell)e^{i\ell I+ip(M+1)}=e^{ip+i\ell I}+S(p,\ell)e^{i\ell+ipI}
$$

which can be simplified to

$$
e^{i(p-\ell)(I-1)+i\ell M}+S(p,\ell)e^{ipM}=1+S(p,\ell)e^{i(p-\ell)(I-1)}\,.
$$

This equation must hold for any value of I. Simple algebra gives the two conditions

$$
S=e^{i\ell M}\,, \qquad e^{i(p+\ell)M}=1\longrightarrow p+\ell=\frac{2\pi j}{M}\,. \tag{B.6}
$$

This latter quantization of $p+\ell$ can also be seen as a result of invariance of (B.1) and (B.1) under translating both variables $(I,J)\to(I+M,J+M)$.

The Bethe equation (B.5) becomes

$$\left(\cos\frac{\pi j}{M} - e^{i\frac{(p-\ell)}{2}}\right) + (-1)^j \left(\cos\frac{\pi j}{M} - e^{-i\frac{(p-l)}{2}}\right) e^{i\frac{(p-l)}{2}(2-M)} = 0\,. \tag{B.7}$$

This equation is easily solved in the large M limit, which is the case of most interest to us. The result is

$$\frac{p+\ell}{2} = \frac{\pi j}{M}\frac{p-\ell}{2} = \frac{\pi n}{M} + \mathcal{O}\left(\frac{1}{M^2}\right) \tag{B.8}$$

where j and n are either both even or both odd integers. Their sum or difference are therefore always even and are equal to two times any integer. Thus, when M is large,

$$(p,\ell) = \frac{2\pi}{M}(r,s) + \mathcal{O}\left(\frac{1}{M}\right) \qquad r,s \in \mathcal{Z}$$

and

$$S = 1 + \mathcal{O}\left(\frac{1}{M}\right).$$

Thus, we see that, in the large M limit, the extension of the function $\mathcal{O}_{IJ}$, which was defined only for $I \leq J$, to all I and J, is as a symmetric doubly periodic function. In this limit, the spectrum of the hamiltonian is

$$\frac{g_{YM}^2 N}{2M}\left(r^2+s^2\right). \tag{B.9}$$

Note that this matches the string spectrum in the low energy limit, for the state which contains two oscillators, created from the sigma model vacuum by $a_r^\dagger$ and $a_s^\dagger$ and which has one unit of light-cone momentum, $k=1$.

Now, we consider the operator where two impurities are inserted into k chains $A_1 \ldots A_M$. It can have the form

$$\mathcal{O}_{IJ} = \mathrm{TR}\left(A_1 \ldots \Phi_I \ldots \Phi_J \ldots A_M (A_1 \ldots A_M)^{k-1}\right) \tag{B.10}$$

or

$$\mathcal{O}_{IJ} = \mathrm{TR}\left(A_1 \ldots \Phi_I \ldots A_M (A_1 \ldots A_M)^{k'} A_1 \ldots \Phi_J \ldots A_M (A_1 \ldots A_M)^{k-k'-2}\right). \tag{B.11}$$

We can simply treat these as a function where $I \leq J$ and both I and J run from 1 to kM. Then, the previous discussion of the case where $k=1$ can be applied here with the only difference that the integers (r,s) are replaced by the ratios $\left(\frac{r}{k},\frac{s}{k}\right)$. The spectrum of the hamiltonian is given by

$$\frac{g_{YM}^2 N}{2Mk^2}\left(r^2+s^2\right). \tag{B.12}$$

This matches the noninteracting string spectrum when there are k units of light-cone momentum. However, all values of r and s are not allowed, but are restricted by a further condition. To see this, note that we have not yet taken into account that the operator $\mathcal{O}_{IJ}$ is periodic under the shift $(I,J) \to (I+M, J+M)$, this periodicity follows from cyclicity of the trace. We see that the wavefunction transforms as

$$\psi_{I+MJ+M} = \psi_{IJ} e^{2\pi i\left(\frac{r}{k}+\frac{s}{k}\right)}$$

In order for the wave-functions to have the correct periodicity, it is necessary that $\frac{r}{k}+\frac{s}{k}$ is an integer. For this to be the case, r and s must satisfy the further condition $r+s=km$ where m is an integer. It is this integer, m, which is identified with the wrapping number in string theory and the condition that $r+s=km$ then coincides with the level-matching condition, which poses the same restriction on string theory states.

The eigenstates of the hamiltonian are to be used as follows. First, it is straightforward to show that the contact term in the hamiltonian is precisely what is needed to make the hamiltonian hermitean. It does this by cancelling boundary effects in summations by parts in the inner product, so that

$$\langle\psi_1|H_0\psi_2\rangle = \langle H_0\psi_1|\psi_2\rangle$$

where

$$\langle\psi_1|\psi_2\rangle = \sum_{I\leq J=1}^{kM} \psi^\dagger_{1IJ}\psi_{2IJ}\,.$$

The operator $\mathcal{O}_{IJ}$ can be written as a superposition of eigenstates as

$$\mathcal{O}_{IJ} = \sum_E \psi_{IJ}(E)\mathcal{O}(E)$$

and

$$\mathcal{O}(E) = \frac{2}{kM(kM+1)}\sum_{I\leq J=1}^{kM} \psi^\dagger_{IJ}(E)\mathcal{O}_{IJ}\,.$$

C. The $k=3$ wavefunctions $w(x,y)$, $v(x,y)$ and $\psi_\pm(x,y)$

In this appendix we first provide the explicit form of the $k=3$ states $w(x,y)$ and $v(x,y)$ that are gotten by the repeated action of the operator $(D-3M-2)$ on the general linear combination $u(x,y)$, (7.6), of the 9 basis states (7.1). The action of $(D-3M-2)$ on u, w and v is given in (7.7).

$$\begin{aligned}
w(x,y) \,=\, &\theta(y-x)\Big[(c_5-c_4)O^{S3}(x,y)-(c_5+4c_6)O^{S3}(y,x)+(c_4+4c_6)f^{C3}(x,y)-\\
&\quad-(c_1-c_3-3c_7)f^{C21}(x,y)+(c_1-c_2-3c_7)O^{S21}(x,y)+\\
&\quad+(c_3-c_2)D^{12}(x,y)+(c_4-c_5)T^{111}(x,y)\Big]+\\
&+\theta(x-y)\Big[-(c_5+4c_6)O^{S3}(x,y)+(c_5-c_4)O^{S3}(y,x)+\\
&\quad+(c_4+4c_6)f^{C3}(x,y)-(c_1-c_3-3c_7)f^{C21}(x,y)+\\
&\quad+(c_1-c_2-3c_7)O^{S21}(x,y)+(c_3-c_2)D^{12}(x,y)+\\
&\quad+(c_4-c_5)T^{111}(x,y)\Big]
\end{aligned}\tag{C.1}$$

$$\begin{aligned}
v(x,y) \,=\, &\theta(y-x)\Big[(2c_1-c_3-c_2-6c_7)O^{S3}(x,y)+(5c_2+3c_7-c_1-4c_3)O^{S3}(y,x)+\\
&\quad+(5c_3-4c_2-c_1+3c_7)f^{C3}(x,y)+(5c_4-4c_5+4c_6)f^{C21}(x,y)+\\
&\quad+(5c_5-4c_4+4c_6)O^{S21}(x,y)+(c_4+c_5+8c_6)D^{12}(x,y)+
\end{aligned}$$

$$+ (c_2 + c_3 + 6c_7 - 2c_1) T^{111}(x,y) \Big] +$$
$$+\theta(x-y) \Big[(5c_2 + 3c_7 - c_1 - 4c_3) O^{S3}(x,y) + (2c_1 - c_2 - c_3 - 6c_7) O^{S3}(y,x) +$$
$$+ (5c_3 - 4c_2 - c_1 + 3c_7) f^{C3}(x,y) + (5c_4 - 4c_5 + 4c_6) f^{C21}(x,y) +$$
$$+ (5c_5 - 4c_4 + 4c_6) O^{S21}(x,y) + (c_4 + c_5 + 8c_6) D^{12}(x,y) +$$
$$+ (c_2 + c_3 + 6c_7 - 2c_1) T^{111}(x,y) \Big] . \qquad \text{(C.2)}$$

The states $\psi_\pm(x,y)$ that diagonalize the operator $(D - 3M - 2)$ and that, consequently, have a well defined anomalous dimension, in terms of the original coefficients of $u(x,y)$, (7.6), are given by

$$\psi_+(x,y) = \theta(y-x) \Big[(2c_1 - c_2 - c_3 - 3c_4 + 3c_5 - 6c_7) O^{S3}(x,y) +$$
$$+ (-c_1 + 5c_2 - 4c_3 - 3c_5 - 12c_6 + 3c_7) O^{S3}(y,x) +$$
$$+ (-c_1 - 4c_2 + 5c_3 + 3c_4 + 12c_6 + 3c_7) f^{C3}(x,y) +$$
$$+ (-3c_1 + 3c_3 + 5c_4 - 4c_5 + 4c_6 + 9c_7) f^{C21}(x,y) +$$
$$+ (3c_1 - 3c_2 - 4c_4 + 5c_5 + 4c_6 - 9c_7) O^{S21}(x,y) +$$
$$+ (-3c_2 + 3c_3 + c_4 + c_5 + 8c_6) D^{12}(x,y) +$$
$$+ (-2c_1 + c_2 + c_3 + 3c_4 - 3c_5 + 6c_7) T^{111}(x,y) \Big] +$$
$$+\theta(x-y) \Big[(-c_1 + 5c_2 - 4c_3 - 3c_5 - 12c_6 + 3c_7) O^{S3}(x,y) +$$
$$+ (2c_1 - c_2 - c_3 - 3c_4 + 3c_5 - 6c_7) O^{S3}(y,x) +$$
$$+ (-c_1 - 4c_2 + 5c_3 + 3c_4 + 12c_6 + 3c_7) f^{C3}(x,y) +$$
$$+ (-3c_1 + 3c_3 + 5c_4 - 4c_5 + 4c_6 + 9c_7) f^{C21}(x,y) +$$
$$+ (3c_1 - 3c_2 - 4c_4 + 5c_5 + 4c_6 - 9c_7) O^{S21}(x,y) +$$
$$+ (-3c_2 + 3c_3 + c_4 + c_5 + 8c_6) D^{12}(x,y) +$$
$$+ (-2c_1 + c_2 + c_3 + 3c_4 - 3c_5 + 6c_7) T^{111}(x,y) \Big] \qquad \text{(C.3)}$$

$$\psi_-(x,y) = \theta(y-x) \Big[(-2c_1 + c_2 + c_3 - 3c_4 + 3c_5 + 6c_7) O^{S3}(x,y) +$$
$$+ (c_1 - 5c_2 + 4c_3 - 3c_5 - 12c_6 - 3c_7) O^{S3}(y,x) +$$
$$+ (c_1 + 4c_2 - 5c_3 + 3c_4 + 12c_6 - 3c_7) f^{C3}(x,y) +$$
$$+ (-3c_1 + 3c_3 - 5c_4 + 4c_5 - 4c_6 + 9c_7) f^{C21}(x,y) +$$
$$+ (3c_1 - 3c_2 + 4c_4 - 5c_5 - 4c_6 - 9c_7) O^{S21}(x,y) +$$
$$+ (-3c_2 + 3c_3 - c_4 - c_5 - 8c_6) D^{12}(x,y) +$$
$$+ (2c_1 - c_2 - c_3 + 3c_4 - 3c_5 - 6c_7) T^{111}(x,y) \Big] +$$
$$+\theta(x-y) \Big[(c_1 - 5c_2 + 4c_3 - 3c_5 - 12c_6 - 3c_7) O^{S3}(x,y) +$$
$$+ (-2c_1 + c_2 + c_3 - 3c_4 + 3c_5 + 6c_7) O^{S3}(y,x) +$$
$$+ (c_1 + 4c_2 - 5c_3 + 3c_4 + 12c_6 - 3c_7) f^{C3}(x,y) +$$
$$+ (-3c_1 + 3c_3 - 5c_4 + 4c_5 - 4c_6 + 9c_7) f^{C21}(x,y) +$$

$$
\begin{aligned}
&+ (3c_1 - 3c_2 + 4c_4 - 5c_5 - 4c_6 - 9c_7)O^{S21}(x,y) + \\
&+ (-3c_2 + 3c_3 - c_4 - c_5 - 8c_6)D^{12}(x,y) \\
&+ (2c_1 - c_2 - c_3 + 3c_4 - 3c_5 - 6c_7)T^{111}(x,y)\Big] .
\end{aligned} \tag{C.4}
$$

Denoting by $\psi^i_\pm$ $i = 1, \ldots, 7$ the coefficients of the 7 operators in $\psi_\pm(x,y)$, it is easy to find 5 independent relations between these coefficients both in $\psi_+(x,y)$ and in $\psi_-(x,y)$. For $y \geq x$ one has

$$
\begin{aligned}
\psi^1_\pm + \psi^2_\pm + \psi^3_\pm &= 0 \\
\psi^4_\pm + \psi^5_\pm - \psi^6_\pm &= 0 \\
\psi^1_\pm + \psi^7_\pm &= 0 \\
4\psi^1_\pm - \psi^3_\pm \pm 3\psi^4_\pm &= 0 \\
4\psi^1_\pm - \psi^2_\pm \mp 3\psi^5_\pm &= 0
\end{aligned} \tag{C.5}
$$

for $y \leq x$ it is sufficient to exchange $1 \leftrightarrow 2$ in the above. Therefore the number of independent states of this form is 4. These are the states with period 6 that should have corrections to all orders in the string loop expansion, whereas the other 5 states needed to complete the hamiltonian eigenfunctions in the $k = 3$ case, $u^i(x,y)$ $i = 1, \ldots, 5$, are given in the text and do not have corrections beyond the planar level.

References

[1] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] S.-M. Lee, S. Minwalla, M. Rangamani and N. Seiberg, *Three-point functions of chiral operators in $D = 4$, $N = 4$ SYM at large-N*, *Adv. Theor. Math. Phys.* **2** (1998) 697 [hep-th/9806074].

[5] D.Z. Freedman, S.D. Mathur, A. Matusis and L. Rastelli, *Correlation functions in the CFT(d)/AdS(d + 1) correspondence*, *Nucl. Phys.* **B 546** (1999) 96 [hep-th/9804058].

[6] G. Chalmers, H. Nastase, K. Schalm and R. Siebelink, *R-current correlators in $N = 4$ super Yang-Mills theory from anti-de Sitter supergravity*, *Nucl. Phys.* **B 540** (1999) 247 [hep-th/9805105].

[7] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 582** (2000) 155 [hep-th/0003055].

[8] N. Drukker and D.J. Gross, *An exact prediction of $N = 4$ SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [hep-th/0010274].

[9] G.W. Semenoff and K. Zarembo, *More exact predictions of SUSYM for string theory*, *Nucl. Phys.* **B 616** (2001) 34 [hep-th/0106015].

[10] G.W. Semenoff and K. Zarembo, *Wilson loops in SYM theory: from weak to strong coupling*, *Nucl. Phys.* **108** *(Proc. Suppl.)* (2002) 106 [hep-th/0202156].

[11] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *Penrose limits and maximal supersymmetry*, *Class. and Quant. Grav.* **19** (2002) L87 [hep-th/0201081];
M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, *A new maximally supersymmetric background of IIB superstring theory*, *J. High Energy Phys.* **01** (2002) 047 [hep-th/0110242].

[12] R. Güven, *Plane wave limits and T-duality*, *Phys. Lett.* **B 482** (2000) 255 [hep-th/0005061].

[13] R.R. Metsaev, *Type iib Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044].

[14] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[15] G. 't Hooft, *A planar diagram theory for strong interactions*, *Nucl. Phys.* **B 72** (1974) 461.

[16] D.J. Gross, A. Mikhailov and R. Roiban, *Operators with large R charge in $N = 4$ Yang-Mills theory*, *Ann. Phys. (NY)* **301** (2002) 31 [hep-th/0205066].

[17] C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *A new double-scaling limit of $N = 4$ super Yang-Mills theory and PP-wave strings*, *Nucl. Phys.* **B 643** (2002) 3 [hep-th/0205033].

[18] N.R. Constable et al., *PP-wave string interactions from perturbative Yang-Mills theory*, *J. High Energy Phys.* **07** (2002) 017 [hep-th/0205089].

[19] M. Bianchi, B. Eden, G. Rossi and Y.S. Stanev, *On operator mixing in $N = 4$ SYM*, *Nucl. Phys.* **B 646** (2002) 69 [hep-th/0205321].

[20] N. Beisert, C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *BMN correlators and operator mixing in $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 650** (2003) 125 [hep-th/0208178].

[21] N.R. Constable, D.Z. Freedman, M. Headrick and S. Minwalla, *Operator mixing and the BMN correspondence*, *J. High Energy Phys.* **10** (2002) 068 [hep-th/0209002].

[22] M. Spradlin and A. Volovich, *Superstring interactions in a pp-wave background*, *Phys. Rev.* **D 66** (2002) 086004 [hep-th/0204146].

[23] C.-S. Chu, V.V. Khoze and G. Travaglini, *Three-point functions in $N = 4$ Yang-Mills theory and pp-waves*, *J. High Energy Phys.* **06** (2002) 011 [hep-th/0206005].

[24] M. Spradlin and A. Volovich, *Superstring interactions in a pp-wave background, II*, *J. High Energy Phys.* **01** (2003) 036 [hep-th/0206073].

[25] C.-S. Chu, V.V. Khoze and G. Travaglini, *pp-wave string interactions from n-point correlators of BMN operators*, *J. High Energy Phys.* **09** (2002) 054 [hep-th/0206167].

[26] C.-S. Chu, V.V. Khoze, M. Petrini, R. Russo and A. Tanzini, *A note on string interaction on the pp-wave background*, *Class. and Quant. Grav.* **21** (2004) 1999 [hep-th/0208148].

[27] A. Pankiewicz, *More comments on superstring interactions in the pp-wave background*, *J. High Energy Phys.* **09** (2002) 056 [hep-th/0208209].

[28] A. Pankiewicz and J. Stefanski, B., *pp-wave light-cone superstring field theory*, *Nucl. Phys.* **B 657** (2003) 79 [hep-th/0210246].

[29] C.-S. Chu, M. Petrini, R. Russo and A. Tanzini, *String interactions and discrete symmetries of the pp-wave background*, *Class. Quant. Grav.* **20** (2003) S457–S464 [hep-th/0211188].

[30] Y.-H. He, J.H. Schwarz, M. Spradlin and A. Volovich, *Explicit formulas for neumann coefficients in the plane- wave geometry*, *Phys. Rev.* **D 67** (2003) 086005 [hep-th/0211198].

[31] J. Lucietti, S. Schafer-Nameki and A. Sinha, *On the plane-wave cubic vertex*, *Phys. Rev.* **D 70** (2004) 026005 [hep-th/0402185].

[32] P. Gutjahr and A. Pankiewicz, *New aspects of the BMN correspondence beyond the planar limit*, hep-th/0407098.

[33] L.A. Pando Zayas and D. Vaman, *Strings in RR plane wave background at finite temperature*, *Phys. Rev.* **D 67** (2003) 106006 [hep-th/0208066];
B.R. Greene, K. Schalm and G. Shiu, *On the hagedorn behaviour of pp-wave strings and $N = 4$ SYM theory at finite R-charge density*, *Nucl. Phys.* **B 652** (2003) 105 [hep-th/0208163];
Y. Sugawara, *Thermal amplitudes in DLCQ superstrings on pp-waves*, *Nucl. Phys.* **B 650** (2003) 75 [hep-th/0209145];
R.C. Brower, D.A. Lowe and C.-I. Tan, *Hagedorn transition for strings on pp-waves and tori with chemical potentials*, *Nucl. Phys.* **B 652** (2003) 127 [hep-th/0211201];
G. Grignani, M. Orselli, G.W. Semenoff and D. Trancanelli, *The superstring hagedorn temperature in a pp-wave background*, *J. High Energy Phys.* **06** (2003) 006 [hep-th/0301186];
F. Bigazzi and A.L. Cotrone, *On zero-point energy, stability and hagedorn behavior of type-IIB strings on pp-waves*, *J. High Energy Phys.* **08** (2003) 052 [hep-th/0306102];
O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, *The hagedorn/deconfinement phase transition in weakly coupled large-N gauge theories*, hep-th/0310285.

[34] G. Feverati, P.A. Pearce and F. Ravanini, *Lattice approach to excited tba boundary flows: tricritical ising model*, *Phys. Lett.* **B 534** (2002) 216 [hep-th/0202041].

[35] M. Bertolini, J. de Boer, T. Harmark, E. Imeroni and N.A. Obers, *Gauge theory description of compactified pp-waves*, *J. High Energy Phys.* **01** (2003) 016 [hep-th/0209201].

[36] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167.

[37] M. Bershadsky and A. Johansen, *Large-N limit of orbifold field theories*, *Nucl. Phys.* **B 536** (1998) 141 [hep-th/9803249].

[38] M. Alishahiha and M.M. Sheikh-Jabbari, *The pp-wave limits of orbifolded $AdS_5 \times S^5$*, *Phys. Lett.* **B 535** (2002) 328 [hep-th/0203018].

[39] J.A. Minahan and K. Zarembo, *The bethe-ansatz for $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **03** (2003) 013 [hep-th/0212208].

[40] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, *BMN gauge theory as a quantum mechanical system*, *Phys. Lett.* **B 558** (2003) 229 [hep-th/0212269].

[41] N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 664** (2003) 131 [hep-th/0303060].

[42] S. Dobashi and T. Yoneya, *Resolving the holography in the plane-wave limit of AdS/CFT correspondence*, hep-th/0406225.

[43] S. Lee and R. Russo, *Holographic cubic vertex in the pp-wave*, hep-th/0409261.

JHEP11(2004)053